Multiple criteria decision making: eight concepts of optimality[1]

Milan Zeleny
Graduate School of Business Administration,
GBA 626E, Fordham University at Lincoln Center,
New York, NY 10023, USA
E-mail: zeleny@mary.fordham.edu

Multiple Criteria Decision Making (MCDM) is firmly rooted in an alternative concept of optimality where multiple (rather than single) criteria characterize the notion of "the best" (or optimal), as is prevalent in the areas of economics, engineering, management and business. These are often constrained problems where search for an optimal solution requires some form of evaluating criteria performance tradeoffs. Because there are no tradeoffs along a single criterion, optimality is an essentially multi-criteria concept. In this paper we extend and develop the notion of optimum as a *balance among multiple criteria*. We introduce *eight* different, separate and mutually irreducible optimality concepts in a classification scheme where the traditional single-objective optimality is only a special case. These eight optimality concepts provide a useful initiatory framework for the future MCDM research and applications.

Keywords: Optimality, optimization, optimal design, linear programming, multiple criteria decision making

Milan Zeleny is currently Professor of Management Systems at Fordham University in New York City. He also served on the faculties of the University of South Carolina, Copenhagen School of Economics, European Institute for Advanced Studies in Management (Brussels), and for ten years at Columbia University School of Business. He holds Dipl.Ing. from the Prague School of Economics, and an M.S. and Ph.D. from the University of Rochester. He is listed in *Who's Who* (Marquis editions) in Science and Engineering, in America, in the East, and in the World. He has served on editorial boards of *International Journal of Operations and Quantitative Management; Journal of the International Strategic Management; Operations Research; Computers and Operations Research; Future Generations Computer Systems; Fuzzy Sets and Systems; General Systems Yearbook* and is the Editor-in-Chief of *Human Systems Management*. Also OR/MIS Editor and contributor of five entries in the *International Encyclopedia of Business and Management*. Among Zeleny's books are *Multiple Criteria Decision Making* (McGraw-Hill), *Linear Multiobjective Programming* (Springer-Verlag), *Autopoiesis, Dissipative Structures and Spontaneous Social Orders* (Westview Press), *MCDM-Past Decades and Future Trends* (JAI Press), *Autopoiesis: A Theory of Living Organization* (Elsevier North-Holland), and *Uncertain Prospects Ranking and Portfolio Analysis* (Verlag Anton Hain). Zeleny published over 300 papers and articles, ranging from operations research, cybernetics and general systems, to economics, history of science, total quality management, and simulation of autopoiesis and artificial life (AL).

1. Introduction

Any criterion (measure, yardstick) is characterized by a score (or a range of scores) of performance that is most preferred by the decision maker within a given context. Such contextually most preferred scores are, *if feasible*, also optimal.

The difficulty arises when the most preferred scores are infeasible, i.e., when there are explicit or implicit constraints on decision alternatives or criteria that prevent the achievement of the "most preferred" performance.

When the most preferred (optimum) is infeasible, it can only be approached through maximization or minimization with respect to constraints. When the constraints are fixed and there is only a single criterion, we have a case of simple computation.

If we relax at least some of the constraints or consider multiple criteria then the situation turns multidimensional: tradeoffs emerge, criteria scores have to be balanced and computational maximization or minimization becomes inadequate.

When there is only a single criterion, no matter how comprehensive or complex, its maximization or minimization with respect to constraints is sufficient. When there are multiple criteria, more complex, mul-

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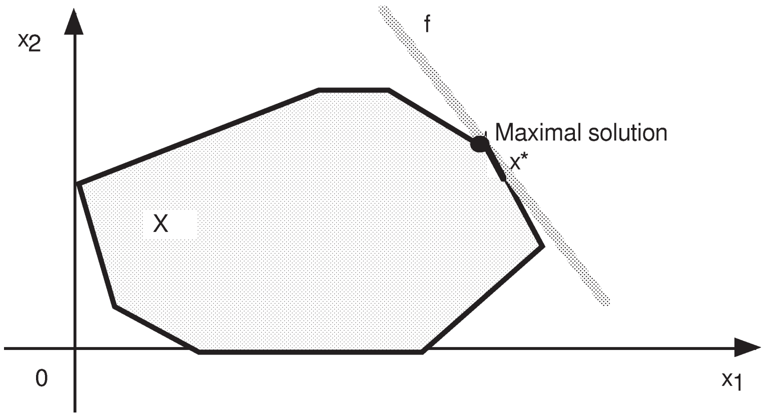

Fig. 1. Single-objective “optimality”.

tiple forms of optimality and optimization have to be explored. In this paper we introduce eight such optimality concepts.

2. Eight concepts of optimality

That, which is given, fixed or determined a priori cannot be subsequently altered and therefore is not a subject of optimization.

What is not given can still to be altered, selected or chosen and it is therefore, by definition, subject to subsequent optimization. Consequently, different optimality concepts can be derived from the distinction between what is given and what is yet to be (optimally) determined.

We have chosen to demonstrate each of the eight concepts in five complementary ways: (a) simple, real-life examples with a defined set of discrete alternatives; (b) formal (mathematical) statement of a general optimization problem; (c) graphical representation of two-dimensional linear-programming problem; and (d) simple numerical problem.

2.1. Single-objective “optimality”

This refers to conventional maximization (or “optimization”). Although not a “tradeoff balancing” problem, it should be included at least for the sake of completeness or as a special case of the *bona fide* optimization.

In order to maximize a single criterion, it is sufficient to perform technical measurement and search tasks. Once X and f were formulated and fixed, the “optimum” (e.g., maximum) is found by computation and no decision or conflict resolution processes are needed. The search for optimality is reduced to *scalarization*, assigning each alternative a number (scalar) and then searching out the highest-numbered alternative.

(1a) From a list of five places (X), find the one that is the cheapest (Min f) for vacations. From a list (X) of affordable (e.g., under $500) transportation modes between L.A. – N.Y.C., find the one that is fastest (Max f), or safest, or cheapest, and so on.

(1b) Formally, we are given a set X of objects-of-choice (alternatives) and a function f, mapping X onto a well-ordered set of real numbers $\mathcal{R}$, i.e., $f: X \to \mathcal{R}$. We then solve Max $f(x)$ subject to $x \in X$. To solve this problem, select $x^* \in X$. If x^* satisfies $f(x^*) \geqslant f(x)$ for all $x \in X$, then f is maximized at x^* and x^* is the maximal solution for $x \in X$.

(1c) Linear-programming maximization problem is:

$$\text{Max}\, f = cx$$
$$\text{s.t. } Ax \leqslant b,\ x \geqslant 0,$$

where $c \in \mathcal{R}^n$ and $A \in \mathcal{R}^{m\times n}$ are coefficient vector and matrix of dimensions $1 \times n$ and $m \times n$, respectively, $b \in \mathcal{R}^m$ is $m \times 1$-dimensional vector of *given* resources and $x \in \mathcal{R}^n$ is $n \times 1$ vector of decision variables (solutions).

In two dimensions (solution x_j, $j = 1, 2$), the objective function f is a straight line, $f = c_1x_1 + c_2x_2$, to be maximized over the convex polygon X of linear inequalities $a_{i1}x_1 + a_{i2}x_2 \leqslant b_i$, $i = 1, 2, \ldots, m$, that is, the intersection of half-planes representing problem constraints. The maximal solution x^* is unique, except under special mathematical conditions.This situation is graphically outlined in Fig. 1.

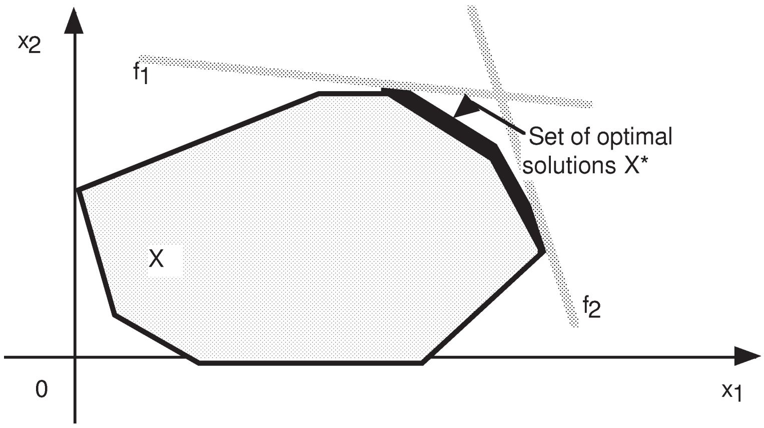

Fig. 2. Multiobjective optimality.

(1d) Consider the following linear-programming problem with two variables and five resource constraints:

$$\text{Max}\, f = 400x_1 + 300x_2$$

subject to

$$\begin{aligned} 4x_1 \quad\quad &\leqslant 20,\\ 2x_1 + 6x_2 &\leqslant 24,\\ 12x_1 + 4x_2 &\leqslant 60,\\ 3x_2 &\leqslant 10.5,\\ 4x_1 + 4x_2 &\leqslant 26. \end{aligned}$$

The maximal solution is $x_1^* = 4.25$, $x_2^* = 2.25$, and $f(x^*) = 2375$. Observe that everything is “given” *a priori* and no market prices of resources are necessary. However, if $p_1 = 30$, $p_2 = 40$, $p_3 = 9.5$, $p_4 = 20$ and $p_5 = 10$ were the respective market prices ($/unit) of the five constrained resources, total cost of the resource portfolio $(20, 24, 60, 10.5, 26)$ would be $B = 2600$.

2.2. Multiobjective optimality

The real optimality, being distinct from above simple computations, involves balancing multiple criteria and their tradeoffs. This corresponds to the vector optimization of Max $f_1(x)$, Max $f_2(x)$, … and Max $f_k(x)$, i.e., simultaneously and subject to $x \in X$.

The concurrent maximization of individual objective functions should be non-scalarized, separate and independent, i.e., not subject to aggregation, like forming and maximizing a superfunction $U\{f_1(x), f_2(x), \dots, f_k(x)\}$. Such aggregation would effectively reduce the multiobjective optimality to a single-objective maximization.

(2a) From a list of five places (X), select the one that is the cheapest (Min f_1) and safest (Max f_2) for vacations. From a list (X) of available (under $500) transportation modes between L.A. – N.Y.C., find the one that is the fastest (Max f_1) and safest (Max f_2) and cheapest (Min f_3), and so on.

(2b) Formally, we are given a set X of alternatives and a set of k functions $f_1, f_2, \dots, f_k$, mapping X onto a well-ordered set of real numbers $\mathcal{R}$, i.e., $f_i : X \to \mathcal{R}$ for all i. Define $F : X \to \mathcal{R}^k$ and $F(x) = \{f_1(x), f_2(x), \dots, f_k(x)\}$.

Next, we seek a subset X^* of X such that, for all $x^* \in X^*$ and all $x \in X - X^*$, the following is true: $F(x^*) >_{\mathcal{B}} F(x)$, where $>_{\mathcal{B}}$ marks an (“balancing”) ordering of $\mathcal{R}^k$, and X^* is the *set of optimal solutions*.

(2c) Multiobjective linear-programming problem is:

$$\text{Max}\, F = Cx$$

$$\text{s.t. } Ax \leqslant b,\ x \geqslant 0,$$

where $C \in \mathcal{R}^{k\times n}$ and $A \in \mathcal{R}^{m\times n}$ are coefficient matrices of dimensions $k \times n$ and $m \times n$, respectively, $b \in \mathcal{R}^m$ is $m \times 1$-dimensional vector of *given* resources and $x \in \mathcal{R}^n$ is $n \times 1$ vector of decision variables (solutions).

In two dimensions (solution x_j, $j = 1, 2$), objective functions f_1 and f_2 (or more) are straight lines, $f_k = c_{k1}x_1 + c_{k2}x_2$, $k = 1, 2$. They are to be maximized over the convex polygon X of linear inequalities $a_{i1}x_1 + a_{i2}x_2 \leqslant b_i$, $i = 1, 2, \dots, m$. The solution set X^* is a *set of nondominated solutions* and the general ordering $>_{\mathcal{B}}$ becomes simply $\geqslant$ (“larger or equal to”). The situation is graphically displayed in Fig. 2.

(2d) Numerical example:

$$\text{Max}\, f_1 = 400x_1 + 300x_2, \quad \text{and}$$
$$f_2 = 300x_1 + 400x_2$$

subject to

$$\begin{aligned} 4x_1 \qquad &\leqslant 20, \\ 2x_1 + 6x_2 &\leqslant 24, \\ 12x_1 + 4x_2 &\leqslant 60, \\ 3x_2 &\leqslant 10.5, \\ 4x_1 + 4x_2 &\leqslant 26. \end{aligned}$$

The maximal solution with respect to f_1 is $x_1^* = 4.25$, $x_2^* = 2.25$, $f_1(4.25, 2.25) = 2375$; maximal solution with respect to f_2 is $x_1^* = 3.75$, $x_2^* = 2.75$, $f_2(3.75, 2.75) = 2225$. The set of optimal (nondominated) solutions X^* includes the two extreme points above and their connecting (feasible) line defined as $4x_1 + 4x_2 = 26$. For example, $0.5(4.25, 2.25) + 0.5(3.75, 2.75) = (4.0, 2.5)$ is another nondominated point on this line. Total cost of the resource portfolio remains at $B = 2600$.

2.3. *Optimal system design: single criterion*

Instead of optimizing a given system with respect to selected criteria, humans often seek to construct an optimal system of decision alternatives (optimal feasible set), designed with respect to given criteria. A single-criterion design is the simplest of all such concepts, producing the best (optimal) set of alternatives X at which a single objective function $f(x)$ is maximized, subject to the cost of design (affordability).

(3a) Design a list of affordable places (X) which assure the cheapest (Min f) vacations. Design a list (X) of affordable (under \$500) transportation modes between L.A. – N.Y.C., which assure the fastest (Max f) mode of transportation.

(3b) Formally, we design a feasible set of alternatives where function f reaches its maximum subject to stipulated cost of the design. We seek (design) a subset X of $\mathcal{U}$ (the universal set) so that for some function $\mathcal{C}: \mathcal{U} \to \mathcal{R}$ we have $\mathcal{C}(X) \leqslant c$, where c is stipulated (maximum) budget or cost of design, and for all $x^* \in X$ and all $x \in \mathcal{U}$, the $f(x^*) \geqslant f(x)$ is true. The level c of $\mathcal{C}(X)$ does not have to be fixed *a priori*.

(3c) Linear-programming optimal-design problem is:

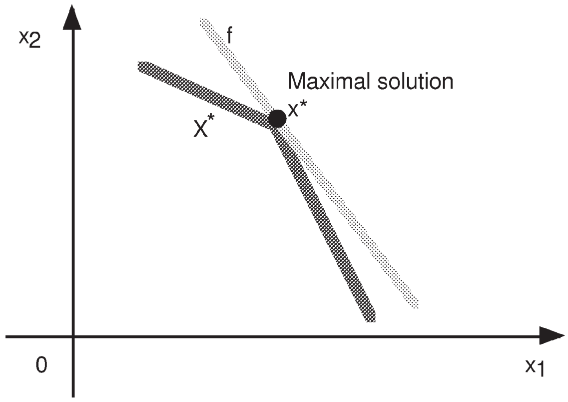

Fig. 3. Optimal system design: single criterion.

$$\begin{aligned} &\text{Max}\, f = cx \\ \text{s.t.}\ &Ax \leqslant b,\ x \geqslant 0, \\ &pb \leqslant B, \end{aligned}$$

where $c \in \mathcal{R}^n$ and $A \in \mathcal{R}^{m\times n}$ are coefficient vector and matrix of dimensions $1 \times n$ and $m \times n$, respectively, $b \in \mathcal{R}^m$ is $m \times 1$-dimensional *unknown* (to be determined) vector of resources and $x \in \mathcal{R}^n$ is $n \times 1$ vector of decision variables (solutions). Further, $p \in \mathcal{R}^m$ is $1 \times m$ vector of unit prices of m resources and B is total available budget (or cost).

Solving the above problem means finding the optimal allocation of budget B so that the resulting (purchased) portfolio of resources b assures the feasibility of x^* and $f(x^*) = \text{Max}\, f(x)$.

In two dimensions (solution x_j, $j = 1, 2$), we are to determine convex polygon X^* by finding affordable b_is for linear inequalities $a_{i1}x_1 + a_{i2}x_2 \leqslant b_i$, $i = 1, 2, \ldots, m$, such that the objective function f is maximized at $x^* \in X^*$. Linear inequalities thus become linear equations and the resulting X^* (their intersection) is reduced to a single point – maximal solution x^*. This situation is graphically represented in Fig. 3.

(3d) Numerical example:

$$\text{Max}\, f = 400x_1 + 300x_2$$

subject to

$$\begin{aligned} 4x_1 \qquad &\leqslant 29.4, \\ 2x_1 + 6x_2 &\leqslant 14.7, \\ 12x_1 + 4x_2 &\leqslant 88.0, \\ 3x_2 &\leqslant 0, \\ 4x_1 + 4x_2 &\leqslant 29.4. \end{aligned}$$

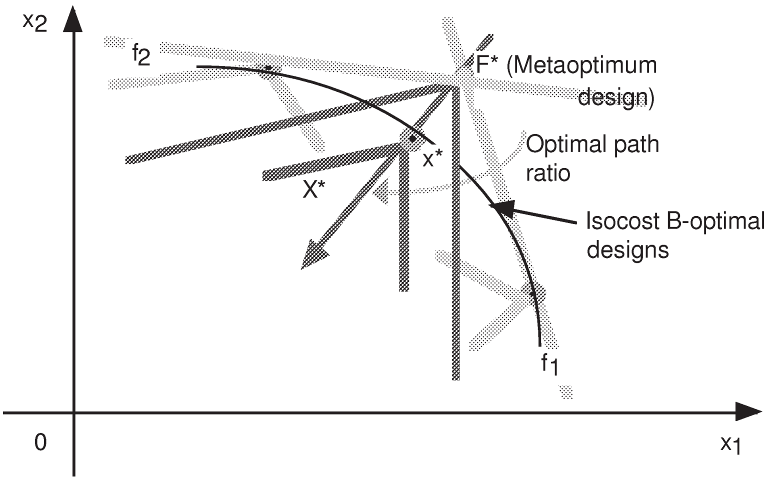

Fig. 4. Optimal system design: multiple criteria.

The right-hand sides (resource portfolio) have been optimally designed. Solving the above optimally designed system will yield $x_1^* = 7.3446$, $x_2^* = 0$, and $f(x^*) = 2937.84$. If market prices of the five resources, $p_1 = 30$, $p_2 = 40$, $p_3 = 9.5$, $p_4 = 20$ and $p_5 = 10$, remain unchanged, the total cost of the resource portfolio $(29.4, 14.7, 88.0, 29.4)$ remains $B = 2600$.

2.4. Optimal system design: multiple criteria

This optimal-system design involves multiple criteria. As before, multiple criteria should not be scalarized into a superfunction U. Rather, all such criteria "compete" independently or there is no need for their separate treatment.

(4a) Design a list of affordable places (X), which assures the cheapest (Min f_1) and the safest (Max f_2) for vacations. Design a list (X) of affordable (under \$500) transportation modes between L.A. – N.Y.C., which assures the fastest (Max f_1) and safest (Max f_2) and cheapest (Min f_3), and so on.

(4b) Formally, we seek to design a subset X of $\mathcal{U}$ so that for some function $\mathcal{C} : \mathcal{U} \to \mathcal{R}$ we have $\mathcal{C}(X) < c$, where c is stipulated (maximum) cost of design, there is a subset X^* of X such that for all $x^* \in X^*$ and all $x \in \mathcal{U}$ the $F(x^*) >_{\mathcal{B}} F(x)$ is true. The $>_{\mathcal{B}}$ is an ("balancing") ordering of $\mathcal{R}^k$ and X^* is the *set of optimal designs*.

(4c) Linear multiobjective-design problem is:

$$\text{Max}\, F = Cx$$
$$\text{s.t. } Ax \leqslant b,\ x \geqslant 0,$$
$$pb \leqslant B,$$

where $C \in \mathcal{R}^{k\times n}$ and $A \in \mathcal{R}^{m\times n}$ are coefficient matrices of dimensions $k \times n$ and $m \times n$, respectively, $b \in \mathcal{R}^m$ is $m \times 1$-dimensional *unknown* (to be determined) vector of resources and $x \in \mathcal{R}^n$ is $n \times 1$ vector of decision variables (solutions). Further, $p \in \mathcal{R}^m$ is $1 \times m$ vector of unit prices of m resources and B is total available budget (or cost).

Solving the above problem means finding the optimal allocation of budget B so that the resulting (purchased) portfolio of resources b assures the feasibility of x^* and $F(x^*)$ at cost B and is "as close as possible" to the metaoptimum design F^* at cost B^*, $B^* \geqslant B$.

In two dimensions (solution x_j, $j = 1, 2$), we are to determine convex polygon X^* by finding affordable b_is for linear inequalities $a_{i1}x_1 + a_{i2}x_2 \leqslant b_i$, $i = 1, 2, \ldots, m$, such that objective functions f_1 and f_2 are straight lines,

$$f_k = c_{k1}x_1 + c_{k2}x_2, \quad k = 1, 2,$$

maximized at $x^* \in X^*$. Linear inequalities become linear equations and resulting X^* (their intersection) is reduced to a single point, maximal solution x^* at cost B. This situation is graphically represented in Fig. 4.

Instead of the set of nondominated solutions, we now have a set (or family) of optimal system designs, characterized by the same cost B and perhaps varying relative importance of objective functions f_1 and f_2.

(4d) Numerical example:

$$\text{Max}\, f_1 = 400x_1 + 300x_2, \quad \text{and}$$
$$f_2 = 300x_1 + 400x_2$$

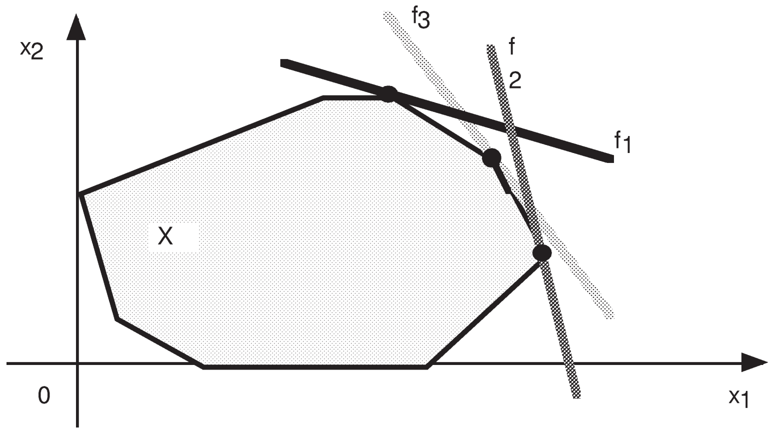

Fig. 5. Single-objective valuation.

subject to

$$\begin{aligned} 4x_1 \quad &\leqslant 16.12, \\ 2x_1 + 6x_2 &\leqslant 23.3, \\ 12x_1 + 4x_2 &\leqslant 58.52, \\ 3x_2 &\leqslant 7.62, \\ 4x_1 + 4x_2 &\leqslant 26.28. \end{aligned}$$

The above is optimally designed portfolio of resources. The maximal solution with respect to both f_1 and f_2 is $x_1^* = 4.03$, $x_2^* = 2.54$, $f_1(4.03, 2.54) = 2375$ and $f_2(4.03, 2.54) = 2225$. This can be compared (for reference only) with the f_1 and f_2 performances in the earlier case of "given" right-hand sides. Assuming the same prices of resources, total cost of this resource portfolio is $B = 2386.74 \leqslant 2600$. One can calculate even better performing portfolios by spending the entire budget of 2600 (i.e., additional 213.26).

2.5. Optimal valuation: single criterion

All previously discussed optimization forms assume that decision criteria are given *a priori*. However, in *human* decision-making, different criteria are continually being tried and applied, some are discarded, new ones added, until an optimal (properly balanced) mix of both quantitative and qualitative criteria is identified. There is nothing more suboptimal than engaging perfectly good means X towards unworthy, ineffective or arbitrarily determined criteria (goals or objectives). Optimizing wrong criteria cannot yield optimal solutions.

If the set of alternatives X is given and fixed *a priori*, we face a problem of optimal *valuation*: According to what measure should the alternatives be evaluated and ordered? According to criterion f_1 or f_2 or f_3? Should most satisfactory vacations be measured by cost? Entertainment value? Privacy conditions? Which of the criteria captures best our values and purposes? What specific criterion engages the available means (X) in the most effective way?

(5a) Select a single criterion (f), perhaps from a list $(f_1, f_2, f_3, \ldots)$, like entertainment, education, privacy, cost, etc., which would best evaluate and order a given list of affordable places (X) to secure most satisfactory or fulfilling (through Max f or Min f) vacations. Select a proper criterion (f) to evaluate a list (X) of affordable (under \$500) transportation modes between L.A. – N.Y.C., which assures the most satisfactory or the most effective (via Max f or Min f) mode of transportation.

(5b) Formally, we define $\mathcal{F}$ to be the set of all functions $f : X \to \mathcal{R}$. Let INT be a subset of $(\mathcal{F} \times \mathcal{U})$ of *integrated problems* in terms of relationships between criteria f and alternatives X. Select a function $f^* \in \mathcal{F}$ such that there is $x^* \in X$ for all $x \in \mathcal{U}$ and all $f \in \mathcal{F}$ such that $f^*(x^*) \geqslant f(x)$, and $(f^*, X) \in$ INT.

Because we have to select the most suitable criterion f according to higher criteria, a metacriterion *value complex* $V[f, X]$ must be invoked.

(5c) Linear-programming valuation problem: Find f such that

$$\text{Max}\, f = cx$$
$$\text{s.t.}\ \ Ax \leqslant b,\ \ x \geqslant 0,$$

provides the best or most effective valuation of X with respect to a given complex of values $V[f, X]$ and assures that optimal pattern (f^*, b, x^*) is preferred to any (f_k^*, b, x_k^*), for all k.

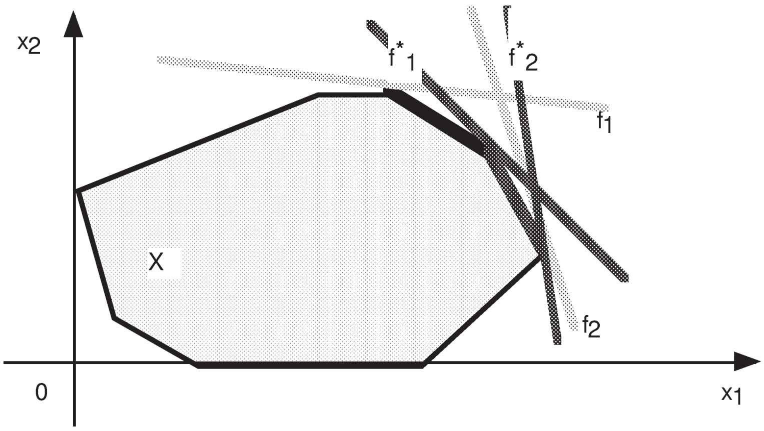

Fig. 6. Multiobjective valuation.

In two dimensions, we are asked to evaluate convex polygon X by selecting an objective function f so that f maximization at $x \in X$ is preferred to f_k maximization for all k. This situation is graphically represented in Fig. 5.

(5d) Numerical example. In order to evaluate X, should we maximize f_1 or f_2? How do we select a criterion if only one is allowed (possible) or feasible?

$$\text{Max } f_1 = 400x_1 + 300x_2 \quad \text{or}$$

$$\text{Max } f_2 = 300x_1 + 400x_2$$

subject to

$$4x_1 \leqslant 20,$$

$$2x_1 + 6x_2 \leqslant 24,$$

$$12x_1 + 4x_2 \leqslant 60,$$

$$3x_2 \leqslant 10.5,$$

$$4x_1 + 4x_2 \leqslant 26.$$

The maximal solution with respect to f_1 is $x_1^* = 4.25$, $x_2^* = 2.25$, $f_1(4.25, 2.25) = 2375$. Maximal solution with respect to f_2 is $x_1^* = 3.75$, $x_2^* = 2.75$, $f_2(3.75, 2.75) = 2225$. Is 2375 of f_1 better than 2225 of f_2? Only one of these valuation schemes can be selected.

2.6. *Optimal valuation: multiple criteria*

If the set of alternatives X is given and fixed *a priori*, but a set of multiple criteria is still to be selected for the evaluation and ordering of X, we have a problem of multiple-criteria valuation: Which *set* of criteria best captures our value complex $V[f, X]$? Is it (f_1 and f_2)? Or (f_3 and f_4)? Or perhaps (f_1 and f_2 and f_3)? Or some other combination?

(6a) Select a combination of criteria, perhaps from a list $(f_1, f_2, f_3, f_4, \ldots)$, like entertainment, education, privacy, cost, etc., which would best evaluate a given list of affordable places (X) in order to secure most satisfactory or fulfilling (e.g., through Max f_1 and Min f_2) vacations. Choose a set of criteria $(f_1, f_2, \ldots, f_k)$ to evaluate a list (X) of affordable (under \$500) transportation modes between L.A. – N.Y.C., assuring the most satisfactory or effective (via Max f_1 and Min f_2) mode of transportation.

(6b) Formally, we define $\mathcal{F}^k$ to be the set of all functions $F: X \to \mathcal{R}^k$, where $F(x) = \{f_1(x), f_2(x), \ldots, f_k(x)\}$. Let MINT be a subset of $(\mathcal{F}^k \times \mathcal{U})$, the set of multiple criteria problems, *integrated* in terms of the relationships between criteria F and alternatives X.

Select a function $F^* \in \mathcal{F}^k$ such that there is $x^* \in X$ for all $x \in \mathcal{U}$ and all $F \in \mathcal{F}^k$ such that $F^*(x^*) >_{\mathcal{B}} F(x)$ and $(F^*, X) \in$ MINT.

Because we have to identify the most suitable criteria F, a metacriterion *value complex* $V[F, X]$ must be used.

(6c) Linear multiobjective valuation problem is: Find F such that

$$\text{Max } F = Cx$$

$$\text{s.t. } Ax \leqslant b, \; x \geqslant 0,$$

provides the best valuation of X with respect to a given $V[F, X]$. Solving this problem means establishing an optimal pattern (F^*, b, x^*).

In two dimensions, we are to evaluate convex polygon X by finding objective functions f_1 and f_2 (straight lines, $f_k = c_{k1}x_1 + c_{k2}x_2$, $k = 1, 2$) so that their maximization at $x \in X$ is preferred to any other pattern. This situation is graphically represented in Fig. 6.

(6d) Numerical example. How do we select a set of criteria, f_1, f_2, or (f_1, f_2), that would best express a given value complex?

$$\text{Max } f_1 = 400x_1 + 300x_2 \quad \text{or/and}$$

$$\text{Max } f_2 = 300x_1 + 400x_2$$

subject to

$$4x_1 \leqslant 20,$$
$$2x_1 + 6x_2 \leqslant 24,$$
$$12x_1 + 4x_2 \leqslant 60,$$
$$3x_2 \leqslant 10.5,$$
$$4x_1 + 4x_2 \leqslant 26.$$

The maximal solution with respect to f_1 is $x_1^* = 4.25$, $x_2^* = 2.25$, $f_1(4.25, 2.25) = 2375$. Maximal solution with respect to f_2 is $x_1^* = 3.75$, $x_2^* = 2.75$, $f_2(3.75, 2.75) = 2225$. Should we use f_1 or f_2, or should we use both f_1 and f_2 to achieve the best valuation of X? Only one of possible (single and multiple criteria) valuation schemes is to be selected.

2.7. *Optimal pattern matching: single criterion*

Human decision making is obviously characterized by a parallel search for both the criteria and alternatives in order to achieve their optimal interaction and thus arrive at an optimal problem statement and its optimal solution. Neither the criteria (f or F), nor the alternatives (X) are "given" *a priori* in a genuine decision-making situation.

There is a problem formulation representing an "optimal pattern" of interaction between alternatives and criteria. It is this optimal, ideal or cognitive-equilibrium problem formulation or pattern that is to be approximated or matched as closely as possible by decision makers. Single-objective matching of the cognitive equilibrium is the simplest special case.

(7a) Select a criterion (f), like entertainment, education, privacy, cost, etc., *and* design an affordable list of places (X), which assures the most satisfactory or fulfilling[2] (through Max f or Min f) vacations. Select a proper criterion (f) and design a list (X) of affordable (under \$500) transportation modes between L.A. – N.Y.C., which assures the most satisfactory (via Max f or Min f) mode of transportation.

(7b) Formally, we define $\mathcal{F}$ to be the set of all functions $f : X \to \mathcal{R}$. Let INT be a subset of $(\mathcal{F} \times \mathcal{U})$ of *integrated problems* in terms of relationships between criteria f and alternatives X.

Select (design) a subset X of $\mathcal{U}$ and function $f^* \in \mathcal{F}$ such that – for some function $\mathcal{C} : \mathcal{U} \to \mathcal{R}$ we have $\mathcal{C}(X) < c$, where c is stipulated (maximum) cost of design – there is $x^* \in X$ for all $x \in \mathcal{U}$ and all $f \in \mathcal{F}$ such that $f^*(x^*) \geqslant f(x)$, and $(f^*, X) \in$ INT.

Because w have to select X and identify the most suitable criterion f, a metacriterion *value complex* $V[f, X]$ must be used.

(7c) Linear single-criterion optimal matching problem is: Find f and b such that

$$\text{Max } f = cx$$
$$\text{s.t. } Ax \leqslant b, \; x \geqslant 0,$$

matches as closely as possible the optimal but infeasible (unaffordable) pattern f^* and b^*, at $pb \leqslant B$.

Solving the above problem means establishing the optimal pattern (f^*, b^*, x^*) and its budgetary level B^*, then finding the optimal allocation of actual budget B so that the resulting (purchased) portfolio of resources b, chosen objective function f and implied solution x assures that pattern (f, b, x) matches the optimal pattern (f^*, b^*, x^*) as closely as possible.

In two dimensions, we are to determine convex polygon X by finding affordable b_is for linear inequalities $a_{i1}x_1 + a_{i2}x_2 \leqslant b_i$, $i = 1, 2, \ldots, m$, and an objective function f so that f maximization at $x \in X$ costs B and it best matches f^* maximized at $x^* \in X^*$, defined by b_i^*s purchased for B^*. The situation is graphically represented in Fig. 7.

(7d) Numerical example. Should we maximize f_1 or f_2? How do we select a single criterion if only one is allowed (possible) or feasible?

$$\text{Max } f_1 = 400x_1 + 300x_2 \quad \text{or}$$

$$\text{Max } f_2 = 300x_1 + 400x_2$$

subject to

$$4x_1 \leqslant 29.4 \quad \text{or} \quad 0,$$
$$2x_1 + 6x_2 \leqslant 14.7 \qquad 41.27,$$
$$12x_1 + 4x_2 \leqslant 88.0 \qquad 27.52,$$
$$3x_2 \leqslant 0 \qquad 20.63,$$
$$4x_1 + 4x_2 \leqslant 29.4 \qquad 27.52.$$

[2] The notion of "fulfilling" or "satisfying" implies the search for balance, harmony or fit between what one wants (or ought) to do and what one actually can do. We choose not only our means but also our objectives and goals to achieve such balance.

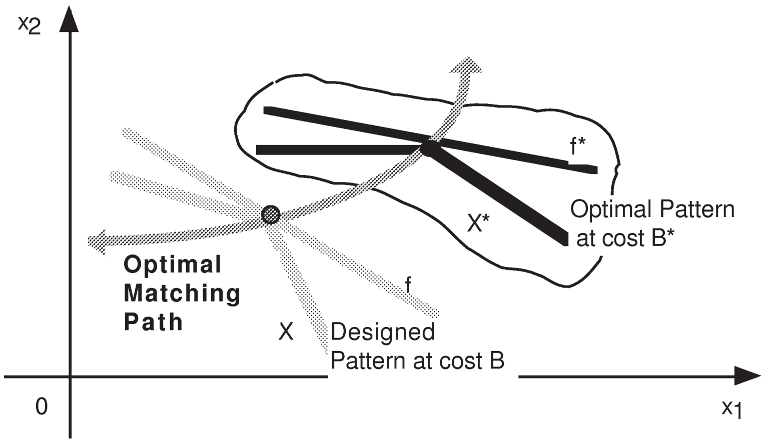

Fig. 7. Optimal pattern matching: single criterion.

The above formulation describes two optimally designed portfolios of resources with respect to f_1 and f_2, respectively. So, among the possible patterns are: (1) $x_1^* = 7.3446$, $x_2^* = 0$; $f_1(x^*) = 2937.84$; $B = 2600$ and (2) $x_1^* = 0$, $x_2^* = 6.8783$; $f_2(x^*) = 2751.32$; $B = 2600$.

Suppose that the value complex implies that the chosen criterion should minimize the loss of the opportunity (i.e., not chosen) criterion, other things being equal. Choosing f_1 would make f_2 drop only to 80.08% of the opportunity performance while choosing f_2 would make f_1 drop to 70.24%. So, f_1 has a preferable opportunity impact and the first pattern and its resource portfolio should be selected.

A value complex indicating that resource quantities used should be as small as possible (not only their prices) would lead to choosing f_2 and thus the second pattern.

2.8. Optimal pattern matching: multiple criteria

Pattern matching with multiple criteria is more involved and so far the most complex optimality concept. In all "matching" concepts there is a need to evaluate the closeness (resemblance or match) of a proposed problem formulation (single- or multi-criterion) to the optimal problem formulation (or pattern).

(8a) Select necessary criteria $(f_1, f_2, \ldots, f_k)$, like entertainment, education, privacy, cost, etc., *and* design an affordable list of places (X), which assure the most satisfying or fulfilling (through Max f_1 and Min f_2) vacations. Select proper criteria $(f_1, f_2, \ldots, f_k)$ *and* design a list (X) of affordable (under \$500) transportation modes between L.A. – N.Y.C., which assure the most satisfactory (via Max f_1 and Min f_2) mode of transportation.

(8b) Formally, we define $\mathcal{F}^k$ to be the set of all functions $F\colon X \to \mathcal{R}^k$, where $F(x) = \{f_1(x), f_2(x), \ldots, f_k(x)\}$. Let MINT be a subset of $(\mathcal{F}^k \times \mathcal{U})$, the set of multiple criteria problems, *integrated* in terms of the relationships between criteria F and alternatives X.

Select (design) a subset X of $\mathcal{U}$ and function $F^* \in \mathcal{F}^k$ such that – for some function $\mathcal{C}\colon\mathcal{U} \to \mathcal{R}$ we have $\mathcal{C}(X) < c$, where c is stipulated (maximum) cost of design – there is $x^* \in X$ for all $x \in \mathcal{U}$ and all $F \in \mathcal{F}^k$ such that $F^*(x^*) >_{\mathcal{B}} F(x)$, and $(F^*, X) \in$ MINT.

Because we have to identify the most suitable criteria F, a metacriterion *value complex* $V[F, X]$ must be used.

(8c) Linear multiobjective pattern matching problem is: Find F and b such that

$$\text{Max}\, F = Cx$$
$$\text{s.t.}\ \ Ax \leqslant b,\ x \geqslant 0,$$

matches as closely as possible optimal but infeasible (unaffordable) pattern F^* and b^*, at $pb \leqslant B$.

Solving the above problem means establishing the optimal pattern (F^*, b^*, x^*) at its budgetary level B^*, then finding the optimal allocation of actual budget B so that the resulting (purchased) portfolio of resources b and chosen set of objective functions F imply solution x, assuring that pattern (F, b, x) matches optimal pattern (F^*, b^*, x^*) as closely as possible.

In two dimensions, we are to determine convex polygon X by finding affordable b_is (X) for linear inequalities $a_{i1}x_1 + a_{i2}x_2 \leqslant b_i$, $i = 1, 2, \ldots, m$, and determine objective functions f_1 and f_2 (straight lines,

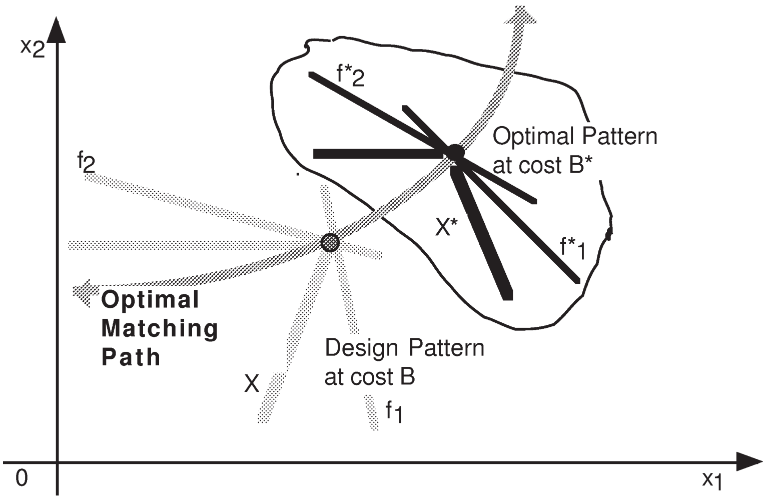

Fig. 8. Optimal pattern matching: multiple criteria.

$f_k = c_{k1}x_1 + c_{k2}x_2$, $k = 1, 2$) so their maximization at $x \in X$, costing B, best matches f_1^* and f_2^* maximized at $x^* \in X^*$, defined by b_i^*s costing B^*. This situation is graphically represented in Fig. 8.

(8d) Numerical example. How do we select a set of criteria, f_1, f_2 or (f_1, f_2), that would best express our value complex?

$$\text{Max } f_1 = 400x_1 + 300x_2 \quad \text{or/and}$$

$$\text{Max } f_2 = 300x_1 + 400x_2$$

subject to

$4x_1 \leqslant 29.4$	or 0	or	19.98,
$2x_1 + 6x_2 \leqslant 14.7$	41.27		28.78,
$12x_1 + 4x_2 \leqslant 88.0$	27.52		72.48,
$3x_2 \leqslant 0$	20.63		9.39,
$4x_1 + 4x_2 \leqslant 29.4$	27.52		32.50.

The above represents three optimally designed portfolios of resources with respect to f_1, f_2 and (f_1, f_2), respectively. So, among the possible patterns are: (1) $x_1^* = 7.3446$, $x_2^* = 0$; $f_1(x^*) = 2937.84$; $B = 2600$, (2) $x_1^* = 0$, $x_2^* = 6.8783$; $f_2(x^*) = 2751.32$; $B = 2600$ and (3) $x_1^* = 4.996$, $x_2^* = 3.131$; $f_1(x^*) = 2937.84$, $f_2(x^*) = 2751.32$; $B = 2951.96$.

If the value complex requires that $B = 2600$ is not exceeded, we may "match" optimal pattern (3) at that level by scaling it down by the optimum-path ratio $r = 2600/2951.96 = 0.88$. The new pattern (3) is: $x_1^* = 4.396$, $x_2^* = 2.755$; $f_1(x^*) = 2585.30$, $f_2(x^*) = 2421.16$; $B = 2600$. If producing both products is valuable, then the choice could be the maximization of both f_1 and f_2, mutatis mutandis.

3. Conclusions

In Table 1 we summarize all eight basic optimality concepts according to a dual classification: single versus multiple criteria compared with the level or extent of the "given", ranging from "all-but" to "none-except". The traditional concept of optimality, characterized by too many "givens" and a single criterion, naturally appears to be the farthest removed from any practical conditions or circumstances for problem solving.

When the decision criteria are fixed *a priori*, like in the first four optimality concepts, their performance achievements provide the measures of goodness. As soon as the criteria themselves are to be selected (like in the last four concepts), then a *value complex* V, or a metacriterion, is presumed and implied. In order to avoid logically infinite regress (criteria for selecting criteria for selecting criteria ...), value complex V must be anchored and integrated in *fundamental values* that are broadly (at least temporarily) accepted and not subject to choice (and thus optimization).

Should we design for speed or for safety? Should we strive for both? Or should we aim at performance, mileage and cost? How do we select the criteria themselves? Usually we take them as being given and thus avoid the problem. Yet, they are not given because somebody did have to select them first. What criteria were used for selecting criteria?

Value complex determines that when we are in no hurry and have children we might opt for safety, while having no children, combined with a sense of urgency or fear, could make us go for speed. Similarly,

Table 1
Eight concepts of optimality

Given	Number of criteria: Single	Multiple
Criteria & alternatives	Traditional "optimality"	MCDM
Criteria only	Optimal design (De Novo programming)	Optimal design (De Novo programming)
Alternatives only	Optimal valuation (Limited equilibrium)	Optimal valuation (Limited equilibrium)
"Value complex" only	Cognitive equilibrium (Matching)	Cognitive equilibrium (Matching)

the choice between profits, quality and costs may be guided by relying on standards of behavior and values like not cheating the customer, serving the public or satisfying the shareholder. Candor, trust, proper recognition and reward could be other useful components of the requisite value complex.

Value complex is based on principles but also rooted in context or circumstances. Value complex consists of mostly qualitative and difficult to measure principles, ethics and rules, expressible only in imprecise and fuzzy language rather than in crisp and quantifiable functions.

There is a difference between constraints, goals and objectives that is merely technical. Imposing a limit, bound or value on an objective will turn it into a goal if approachable by degrees, or into a constraint if it is to be strictly satisfied or adhered to. However, constraints are always stated individually and only rarely (for purely technical reasons) summed up into meaningless aggregates or "superconstraints".

Yet, the same economic variables, the same "constraints" are freely combined, scalarized, added up, weighted and aggregated as soon as their constraining values are relaxed or made approachable, as soon as they become goals or objectives. This represents a profound dilemma: how can criteria become additive and constraints not, even when they may have identical economic interpretations and contents?

To answer this dilemma, it has been proposed here to broaden the traditional concept of optimality into the eight different and mutually irreducible forms.

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