*Session Notes for Interest Rate Derivatives*
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In the absence of arbitrage, purely technical conditions (see Duffie Chapter 6) are required for the existence of an equivalent martingale measure. Such a probability measure has the property that any security with a finite-variance payoff Z at time s has a price, at any time $t <= s$, of

$$\Lambda_{t,s} = E_t^Q \left[ exp\left(\int_t^s -r_u du\right) Z_s(r_t, ..., r_s, s) \right] \tag{1}$$

The measure $Q$ is often called the pricing measure or the risk neutral measure. The process $\Lambda$ is sometimes known as the discount function or loosely the term structure of interest rates. The term structure is often expressed in terms of the yield curve. The continuously compounded yield $y_{t,\tau}$ on a zero coupon bond maturing at time $t + \tau$ is defined by

$$y_{t,\tau} = -log(\Lambda_t, t + \tau)/\tau \tag{2}$$

Of course when you know $y_{t,\tau}$ (yield), you can go back to price:

$$\Lambda_{t,t+\tau} = exp(-\tau \times y_{t,\tau}) \tag{3}$$

## One factor term structure models

By one factor term structure models we mean models of the short rate r given by a stochastic differential equation of the form

$$dr_t = \mu_{r_t,t} dt + \sigma(r_t, t) d\hat{B}_t \tag{4}$$

where we place technical conditions on $\mu$ and $\sigma$ to guarantee equation 3 results in a finite and well-defined $\Lambda_{t,t+\tau}$.

While it is possible to model make $\sigma(r_t, t)$ say a 2 by 1 vector such that $\sigma(r_t, t)d\hat{B}_t$ can really be $\sigma_1(r_t, t)d\hat{B1}_t + \sigma_2(r_t, t)d\hat{B2}_t$,for now we will consider cases where $d\hat{B}_t$ is a scalar (one random variable).

One factor models are named one-factor not because $d\hat{B}_t$ is one dimensional but because equation 4 means that at any time the yield curve for all maturities is completely determined by $r$ (a scalar). In other words, given functions $\mu$ and $\sigma$ the level of $r$ determines the entire yield curve.

Equation 4, means that $r$ under $Q$ measure is markovian, which means that knowledge of current r $r_t$ and functions $\mu$ and $\sigma$ completely characterizes $r_s$ for any $s >= t$. Past values of r, i.e. $r_s, s < t$ do not help predict the future of r.

| Model | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\beta_1$ | $\beta_2$ | $\nu$ |
|---|---|---|---|---|---|---|
| Cox-Ingersoll-Ross | X | X |  |  | X | 0.5 |
| Pearson-Sun | X | X |  | X | X | 0.5 |
| Dothan |  |  |  |  | X | 1 |
| Brennan-Schwartz | X | X |  |  | X | 1 |
| Merton (Ho-Lee) | X |  |  | X |  | 1 |
| Vasicek | X | X |  | X |  | 1 |
| Black-Karasinski |  | X | X |  | X | 1 |
| Constantinides-Ingersoll |  |  |  |  | X | 1.5 |

Table 1. Common Single Factor Model Parameters: X's denote the non-zero parameters in a model, Copyright Duffie, 1996

In the finance literature several one-factor models have been investigated. Many of these are nested in a special case of 4.

$$dr_t = [\alpha_1(t) + \alpha_2(t)r_t + \alpha_3(t)r_t * log(r_t)]dt + [\beta_1(t) + \beta_2(t)r_t]^\nu d\hat{B}_t \tag{5}$$

for continuous functions $\alpha_1$, $\alpha_2$,$\alpha_3$, $\beta_1$, $\beta_2$ and for $0.5 <= \nu <= 1.5$.

The below table shows exactly how several models in the literature are nested in this specification. The $X$'s denote non-zero parameter in the model.

There are two main methods to solve for the bond yields for a given maturity,T, at time t. (i.e. calculate $\Lambda_t, T$

1) The integral in Equation 1 can be computed numerically or analytically.

2) The Feynman-Kac approach where one exploits the below mathematical fact. Defining

$$\Lambda_{t,T} = f(r_t, t); \tag{6}$$

Then, we can analytically or numerically solve the below partial differential equation and its associated terminal condition to obtain $f(r_t, t)$. The partial differential equation is,

$$f_t(x,t) + f_x(x,t)\mu_(x,t) + (1/2)f_{xx}(x,t)\sigma(x,t)^2 - xf(x,t) = 0. \tag{7}$$

The terminal condition is

$$f(x,T) = 1 \tag{8}$$

which simply states that at expiration the bond price is equal to the face value.

**Cox-Ingersoll-Ross Model**

One of the best known single factor term structure models is the Cox-Ingersoll-Ross (CIR) model. In this model $\mu(x,t) = A(\bar{x} - x)$ and $\sigma(x,t) = C\sqrt(x)$ for $x >= 0$ for constants $A, \bar{x}$ and $C$. So the stochastic differential equation describing the short rate under measure $Q$, becomes

$$dr_t = A(\bar{x} - r_t)dt + C\sqrt{r_t}d\hat{B} \tag{9}$$

Lets apply the Feynman-Kac approach to the Cox-Ingersoll-Ross model.

Again, fixing maturity at T, we define $f(r_t, t) = \lambda_{t,T}$, i.e. the price now at time t of a zero coupon bond that matures at time T.

The differential equation for price is

$$f_t(x,t) + f_x(x,t) \times A(\bar{x} - x) + 1/2 f_{xx}(x,t)C^2 x - x f(x,t) = 0 \tag{10}$$

As usual, the terminal condition is

$$f(x,T) = 1 \tag{11}$$

.

Now this is a cauchy type pde and we can check by direct differentiation that $f$ given below solves the pde and satisfies the terminal boundary condition.

$$f(x,t) = H_1(T-t) exp[-H_2(T-t)x] \tag{12}$$

where

$$H_1(t) = \left[\frac{2\gamma e^{(\gamma+A)t/2}}{(\gamma+A)(e^{\gamma t}-1)+2\gamma}\right]^{2A\bar{x}/C^2} \tag{13}$$

and

$$H_2(t) = \frac{2(e^{\gamma t - 1}}{(\gamma+A)(e^{\gamma t}-1)+2\gamma} \tag{14}$$

and

$$\gamma = \sqrt{A^2 + 2C^2} \tag{15}$$

Recalling

$$\Lambda_{t,T} = f(r_t, t); \tag{16}$$

we can write

$$\Lambda_{t,s} = H_1(s-t) exp[-H_2(s-t)r_t] \tag{17}$$

**Matlab Implementation of CIR**

Lets fix A = 0.5, $\bar{x}$ = 0.0719 and C = 0.1664 and the current short rate $r_t$ at 7.5% and write a program to compute zero coupon bond prices and yields. cir.m does this. How do we interpret 7.5% as $r_t$? This could be the per-annum-yield on a short maturity T-bill.

How do we take the CIR model to data? One way is to pick $A$, $\bar{x}$ and $C$ such that the model does a good job in pricing say 2,3,4,5 year bonds historically given the short rate (say the 1 year bond yield). A useful data set here is the Fama-Bliss dataset (famabliss.txt). First column is the day of the measurement. The next five columns are the monthly observations of percent yields (not log) of 1-5 year zero coupon bonds, inferred from the treasury yield curve. Taking $r_t$ as the 1-year log yield, we can find implied yields for 2,3,4,5 year maturities. We can optimize over $A$, $\bar{x}$ and $C$, so that sum of squared differences between actual 2,3,4,5 year yields and model implied yields is minimum (sum taken across months and maturities). Of course, if one would like to emphasize certain maturities or certain dates whose errors could be penalized more in the objective function forcing the model do better there. ciropt.m does the equally weighted optimization.

**The Vasicek Model**

In this model the short rate dynamics under the risk neutral measure is given by;

$$dr = A(\bar{x} - r_t)dt + \sigma d\hat{B}_t \tag{18}$$

Lets define,

$$\Lambda_{t,T} = f(r_t, t); \tag{19}$$

where $\Lambda_{t,T}$ is the price of a zero coupon bond at time t maturing at time T.

Applying the Feynman-Kac approach we obtain the pricing pde.

$$f_t(x,t) + f_x(x,t) \times A(\bar{x} - x) + 1/2 f_{xx}(x,t)\sigma^2 - xf(x,t) = 0 \tag{20}$$

and as usual the terminal condition is

$$f(x,T) = 1 \tag{21}$$

Once again it can be verified by direct differentiation that the solution, $f(x,t)$ is given by

$$f(x,t) = exp(-H_1(T-t) - H_2(T-t)x) \tag{22}$$

where

$$H_2(t) = (1/A) \times (1 - e^{-at}) \tag{23}$$

$$H_1(t) = (t - H_2(t))\bar{x} - (1/2)\nu(t) \tag{24}$$

$$\nu(t) = (\sigma^2)/(2A^3) \times (4e^{-at} - e^{-2at} + 2at - 3) \tag{25}$$

**Matlab Implementation of Vasicek Model**

Let's fix $\bar{x} = 0.07$, $A = 0.18$ and $\sigma = 0.02$ and write a program to compute zero coupon bond prices and yields. vasicek.m does this. Again, we assume that current short rate is 7.5%? Again this could be the per-annum-yield on a short maturity T-bill.

How do we take the Vasicek Model to data? One way is to pick $A$, $\bar{x}$ and $\sigma$ so that the model does a good job pricing say 2,3,4,5 year bonds historically given the short rate (say the 1 year bond yield). We again resort to the Fama-Bliss data set. We apply the same procedure as we did for the CIR model here to pick $A$, $\bar{x}$ and $\sigma$. vasicekopt.m does the equal-weighted optimization.

**Gaussian Single Factor Term Structure Models**

When $\alpha_3 = \beta_2 = 0$ in specification 5, the short rates $r(t_1), \dots, r(t_k)$ for any finite set $t_1, \dots, t_k$ has a joint normal distribution under Q. So the Vasicek Model is a gaussian model and CIR model is not. A more important heavily used gaussian single factor model is the Merton or the Ho-Lee model.

**Merton (Ho-Lee) Model**

In this model the short rate dynamics under the risk neutral measure is given by;

$$dr = \alpha dt + \sigma d\hat{B}_t \tag{26}$$

Defining $\Lambda_{t,T} = f(r_t, t)$, lets apply the Feynman-Kac approach to obtain the pricing pde.

Substituting the drift, $\alpha$and the volatility $\sigma$ terms into equation 7 we obtain

$$f_t(x,t) + f_x(x,t)\alpha + (1/2)f_{xx}\sigma^2 - xf(x,t) = 0 \tag{27}$$

subject to the terminal condition

$$f(x,T) = 1; \tag{28}$$

We can check by direct differentiation that $f$ given below solves the pde and satisfies the terminal boundary condition.

$$f(x,t) = exp(-H_1(T-t) - x(T-t)) \tag{29}$$

where

$$H_1(t) = (1/2)\alpha t^2 - \sigma^2 t^3/6 \tag{30}$$

Now the model is simple enough that we can try some simple stochastic calculus machinery to analytically integrate 1.

Lets start by obtaining an expression for $r_u$, for $u > t$. This requires integrating the equation 26.

The equation is straight forward to integrate;

$$\int_t^s dr = \int_t^s \alpha du + \int_t^s \sigma d\hat{B}_u \tag{31}$$

Yielding;

$$r_s - r_t = \alpha(s-t) + \sigma\hat{W}(s-t) \tag{32}$$

where $\hat{W}$ is a standard brownian motion under Q. This means that conditional on t, $W(s-t)$ is a normal random variable with mean 0 and variance s-t.

Of course, s, in equation 32 is a dummy variable so the equation is really the same as;

$$r_u = r_t + \alpha(u-t) + \sigma\hat{W}(u-t) \tag{33}$$

for $u > s$.

Now lets rewrite the $\int_t^s -r_u du$ substituting in equation 33.

We obtain

$$\int_t^T r_u du = \int_t^T r_t + \alpha(u-t) + \sigma\hat{W(u)} - \hat{W(t)}du \tag{34}$$

. We can integrate each term separately, so we write,

$$\int_t^T r_u du = \int_t^T r_t du + \int_t^T \alpha(u-t)du + \int_t^T \sigma\hat{W(u)}du - \int_t^T \sigma\hat{W(t)}du \tag{35}$$

Now,

$$\int_t^T r_t du = (T-t) * r_t \tag{36}$$

since $r_t$ is known for any $u > t$ . Also,

$$\int_t^T \alpha(u-t)du = \alpha \times (T-t)^2/2 \tag{37}$$

The above is an ordinary calculus integral.

Further,

$$\int_t^T \sigma\hat{W(t)}du = \sigma\hat{W(t)}(T-t) \tag{38}$$

Since $\hat{W(t)}$ is known and therefore a constant for $u > t$.

So the only term here that is a bit of challenge is $\int_t^T \sigma\hat{W(u)}du$.

It is easy to show by Ito's lemma (see Oksendal page 45) that

if $Y_t = t\hat{W}(t)$ then $dY_t = \hat{W}_t dt + td\hat{W}_t$ or in integral form

$$\int_0^t ud\hat{W}u = t\hat{W}(t) - \int_0^t \hat{W}_u du \tag{39}$$

We can write the above equation for $T$.

$$\int_0^T ud\hat{W}u = T\hat{W}(T) - \int_0^T \hat{W}_u du \tag{40}$$

. Subtracting the first equation from the second we obtain;

$$\int_t^T ud\hat{W}u = T\hat{W}(T) - t\hat{W}(t) - \int_t^T \hat{W}(u)du \tag{41}$$

rearranging,

$$\int_t^T \hat{W}(u)du = T\hat{W}(T) - t\hat{W}(t) - \int_t^T ud\hat{W}u \tag{42}$$

We note that $T\hat{W}(T) - t\hat{W}(t) = T\int_t^T d\hat{W}(u)$. Therefore

$$\int_t^T \hat{W}(u)du = \int_t^T (T-u)d\hat{W}u \tag{43}$$

Notice that $E_t(\int_t^T (T-u)d\hat{W}u) = 0$ and further that $Var_t(\int_t^T (T-u)d\hat{W}u) = E_t\left[\int_t^T (T-u)^2 du\right] = (T-t)^3/3$. Further this integral is a normal random variable (it is performing a weighted sum of normal increments).

So, we obtain that

$$\int_t^T r_u du = (T-t)r_t + \alpha(T-t)^2/2 + (T-t)\sigma\hat{W}(t) + \int_t^T (T-u)d\hat{W}u \tag{44}$$

We know that if x is normal with mean $\mu$ and standard deviation $\sigma$, then $E(exp(x)) = exp(\mu + 1/2\sigma^2)$;

Therefore,

$$\Lambda_{t,T} = E_t^Q\left[exp(\int_t^T -r_u du)\right] = exp(-(T-t)r_t - \alpha(T-t)^2/2 + 1/2 \times \sigma^2 \times (T-t)^2/3) \tag{45}$$

which is the same as equation 29 and 33.

Lets investigate some features of the Ho-Lee model.

$\lim_{t\to T}\Lambda_{t,T} = 1$. The value of the bond is approaching face value as maturity goes to zero. The model "pulls to par".

Since, given $r_t$,$r_u, u > t$ is normally distributed (see equation 33), $\Lambda_{u,T}$ is log-normally distributed given $r_t$ under the risk neutral measure.

**Matlab Implementation of Merton (Ho-Lee) Model**

Lets fix $\alpha = 0.01$ and $\sigma = 0.03$ and write a program to compute zero coupon bond prices and yields. mertonholee.m does this.

**Matlab Implementation of the Ho-Lee Model.**

How do we take the Merton (Ho-Lee) Model to data? One way is to pick $\alpha$ and $\sigma$ so that the model does a good job pricing say 2,3,4,5 year bonds historically given the historical short rate.mertonholeeopt.m does the equal weighted optimization (the objective function emphasizes all horizons and date of observations equally).

**Monte-Carlo Pricing of Merton (Ho-Lee) Model.**

We can also try and solve the Merton (Ho-Lee) model using simulations.Lets fix $r_t$ at 7.5% $\alpha = 0.01$ and $\sigma = 0.03$ and price a 2year zero coupon bond.

The equation for short-rate under the risk neutral measure is;

$$dr = 0.01dt + 0.03d\hat{W}t \tag{46}$$

This equation could be simulated forward with the Euler-Scheme. Euler scheme replaces differentials by first differences therefore may end-up causing significant cumulative errors.

$$r_{t+\Delta} - r_t = 0.01\Delta + 0.03 \times N(0, \Delta) \tag{47}$$

where $N(0, \Delta)$ is a normal random variable with zero mean and $\Delta$ variance.

One path of $r$ series is generated by simulating equation 47 forward. This path can be summed across time then multiplied by $\Delta$. Then the associated $\Lambda_t^T$ can be calculated by exponentiating negative of the scaled sum. This procedure can be repeated for say a 1000 times and resulting $\Lambda's$ can be averaged. This average is the price of the bond in question. simmertonholee.m performs the simulation based pricing. Here one can check that the analytical and simulation based methods give almost the same bond prices and difference goes to zero as number of paths and steps become larger. Simcir.m does the simulation based pricing for a two year bond for the Cox-Ingersoll-Ross model and one can check that the simulation based pricing gives close results to the analytical pricing (cir.m). simvasicek.m does the simulation based pricing for the vasicek model and once can check that simulation results are close to analytical results (vasicek.m).

**The Hull and White Model**

In the simplified Hull-White model the interest rates follows a process of the form:

$$dr_t = (\alpha(t) - ar_t)dt + \sigma d\hat{W}_t \tag{48}$$

where $\alpha$ is a deterministic function of $t$ and $a$ and $\sigma$ are constants.

Defining,

$$\Lambda_{t,T} = f(r_t, t); \tag{49}$$

and recalling the fundamental pricing PDE, equation 7 we can write,

$$f_t(x,t) + f_x(x,t)(\alpha(t) - ax) + 1/2 * f_{xx}(x,t)\sigma^2 - xf(x,t) = 0 \tag{50}$$

and the terminal condition is;

$$f(x,T) = 1 \tag{51}$$

It can be verified by direct differentiation that;

$$f(x,t) = exp(-A(x,t) - B(x,t)x) \tag{52}$$

where

$$A(x,t) = \int_t^T e^{-au} \int_t^u e^{ax}\alpha(x)dxdu - 0.5\nu(T-t); \tag{53}$$

$$B(x,t) = (1/a) \times (1 - e^{-a(T-t)}) \tag{54}$$

and

$$\nu(\tau) = \sigma^2/(2a^3) \times (4e^{-a\tau} - e^{-2a\tau} + 2a\tau - 3) \tag{55}$$

The advantage of the Hull-and White model is that one can match the entire initial term structure of interest rates (yield curve) exactly. This is in contrast to say the Merton (Ho-Lee) model whereone can only exactly match 2 initial yields since there are only two parameters ($\alpha$ and $\sigma$). Hull and White model is also in contrast to one factor CIR model where one can only match 3 initial yields since there are only three parameters ($A, \bar{x}$, and $C$). The flexibility in defining $\alpha(t)$ affords us to match the entire initial yield curve.

**Matlab Implementation of Hull-White Model.**

Instead of leaving $\alpha(t)$ completely flexible, we can parameterize $\alpha(t)$ as $q_1 + q_2 t + q_3 t^2 + q_4 t^3$ if we would like to match the 4 initial yields exactly. We can pick $q_1, q_2, q_3$ and $q_4$ so that pricing equation correctly prices the 2,3,4,5 year bonds exactly.

Lets fix $\sigma = 0.02$ and $a = 0.18$. hullwhite.m finds $q_1, q_2, q_3, q_4$ so that bond prices at 2,3,4,5 year maturities at the first date of the Fama-Bliss data set is exactly matched. Armed with this calibration that matches the entire initial yield curve, we can look at model's performance at pricing bonds at other dates.

**The Continuous-time Ho-Lee Model**

In the Ho-Lee model, or rather the continuous time version of the Ho-Lee model the interest rate follows a process of the form

$$dr_t = \alpha(t)dt + \sigma d\hat{B}_t \tag{56}$$

where $\alpha$ is a deterministic function of t and $\sigma$ is a constant. It is the Merton(Ho-Lee) model where $\alpha$ is now allowed to be a deterministic function of time rather than constrained to a constant. Much like the Hull-White model the flexibility in defining $\alpha$ will allow us to match the entire initial yield curve.

Direct integration of equation 56 means that for any s¡t,

$$r_t = r_s + \int_s^t \alpha(u)du + \sigma\hat{W}(t) - \hat{W}(s) \tag{57}$$

This means that conditional on $r_s$, $r_t$ is normal with mean $r_s + \int_s^t \alpha(u)du$ and variance $t - s$.

Let's try and obtain bond pricing in Continuous Ho-Lee model by direct analytic stochastic integration of equation 1.

Lets fix current time at $t$ and the maturity of the bond at time $T$ so consider $\Lambda_t^T$

Now,

$$\int_t^T r_u du = \int_t^T r_t du + \int_t^T \int_t^u \alpha(s) ds du + \sigma \int_t^T \hat{W}(u) du - \int_t^T \hat{W}(t) du \tag{58}$$

Integrating term by term, we can write

$$\int_t^T r_u du = (T-t) * r_t + \int_t^T \int_t^u \alpha(s) ds du + \sigma \int_t^T \hat{W}(u) du - \sigma(T-t)W(t) \tag{59}$$

Recall from equation 43 that

$$\int_t^T \hat{W}(u) du = \int_t^T (T-u) d\hat{W} u \tag{60}$$

So,

$$\int_t^T r_u du = (T-t) r_t + \int_t^T \int_t^u \alpha(s) ds du + \sigma \int_t^T (T-u) d\hat{W} u - \sigma(T-t)W(t) \tag{61}$$

It is easy to see that conditional on time $t$, $\int_t^T r_u du$ is normal with mean $(T-t)r_t + \int_t^T \int_t^u \alpha(s) ds du$ and variance $\sigma^2 (T-t)^3/3$. Recalling that if x is normal with mean $\mu$ and variance $\sigma^2$ then, $E(x) = exp(\mu + 0.5\sigma^2)$, we can write,

$$\Lambda_{t,T} = E_t^Q \left[ exp(\int_t^T -r_u du) \right] = exp \left( -(T-t) r_t - \int_t^T \int_t^u \alpha(s) ds du + \sigma^2 (T-t)^3/6 \right) \tag{62}$$

**Matlab Implementation of the Continuous Ho-Lee model**

We can adopt the same approach we used for hull and white. In fact suppose we could like to match the initial yield curve at 5 points. we could parameterize $\alpha_t = q_1 + q_2 t + q_3 t^2 + q_4 t^3 + q_5 t^4$ and pick $q_1, q_2, q_3, q_4, q_5$. But this is not the best way to define $\alpha$ we would like to be able to match the entire yield curve not just a few points. We can consider a better way to choose $\alpha$ function. We first need to define forward and instantaneous forward rates. Fix current time at $t$ andconsider $\tau_1$ and $\tau_2$ where $t < \tau_1 < \tau_2$.

| $t$ | $\tau_1$ | $\tau_2$ |
|---|---|---|
| $+P_{t,\tau_2}$ | $-P_{t,\tau_2}/P_{\tau_1}$ | |
| $-P_{t,\tau_2}$ | | 1 |

Table 2. Cash flows associated with locking in a rate of investment return from $\tau_1$ to $\tau_2$

Above is a table of cash flows for an investor investing in a $(\tau_2 - t)$ period bond and financing the investment by borrowing at $(\tau_1 - t)$ maturity. $P_{t,\tau_1}$ denotes price of a face value 1 bond maturing at time $\tau_1$. Likewise, $P_{t,\tau_2}$ denotes price of a face value 1 bond maturing at time $\tau_2$. Once can see that the investor is able to lock in a log yield to maturity of $-log(P_{t,\tau_2}/P_{t,\tau_1})/(\tau_2 - \tau_1)$ for investing between $\tau_2$ to $\tau_1$. This log yield is called the forward rate.

For any $u > t$, the instantaneous forward rate $forward(t,u)$ is defined as

$$forward(t,u) = \lim_{k \to 0} -log(P_{t,u+k}/P_{t,u})/(k) = -(log(P_{t,u+k}) - log(P_{t,u}))/(k) = -\partial log(P_{t,u})/\partial u \tag{63}$$

. The interpretation of $forward(t, u)$ is the yield to maturity that can be locked at time $t$ for investing at a future time $u$ where the duration of the investment is extremely small.

Let's find an expression for the instantaneous forward rate in the Continuous Ho-Lee model.

In the model;

$$log(P_{t,T}) = \left(-(T-t)r_t - \int_t^T \int_t^u \alpha(s)dsdu + \sigma^2(T-t)^3/6\right) \tag{64}$$

Therefore

$$\partial log(P_{t,T})/\partial T = -r_t - \int_t^T \alpha(s)ds + \sigma^2(T-t)^2/2 \tag{65}$$

and we conclude that

$$forward(t, T) = r_t + \int_t^T \alpha(s)ds - \sigma^2(T-t)^2/2 \tag{66}$$

If we differentiate equation 66 with respect to $T$ we obtain for any $T > t$.

$$\alpha(T) = forward_T(t, T) + \sigma^2(T-t) \tag{67}$$

How do we implement this with the data? We can use some updated data here. Going to the Bloomberg terminal, I type the command IYC1 I25 $< GO >$. This brings up the US Treasury actives and the associated yield curve. One can click on the "pushpin" icon on the top right and drag drop this icon on excel to export yield, maturity and security tickers. Or one can use export command on the top tool bar to export the yield-maturity information or save the yield curve graph as a -gif file. However, conversing with the Bloomberg helpdesk, my understanding is that current MSQF Fordham license does not allow export of numeric data. Therefore, I simply scroll down to the second page and type the maturities and yields to Excel or Matlab.

| Maturity | Yield |
|---|---|
| 1 month | 2.9389 |
| 3 month | 3.2750 |
| 2 Year | 3.1076 |
| 5 year | 3.4752 |
| 10 year | 4.0679 |
| 30 year | 4.4869 |

Table 3. Input from Bloomberg: command IYC1 I25 $< GO >$

These are percent annualized yields. They are not in logs. We can convert them to log yields and use say linear interpolation (interp1) or piecewise cubic hermite interpolating polynomial (pchip) in Matlab to come up with intermediate values say the yield value for 2 months or 4 years. Then come come up with the function $\alpha(t)$. Contholee.m does compute the $\alpha$ function, this m file also verifies that the model prices the entire input term structure exactly.

**Introduction to Tree based pricing**

We can price using all the models we have seen thus far using tree based methods. The idea here is to represent the risk neutral dynamics of the short rate using a tree. The Euler discretization implies that the short rate one step ahead is always normal conditional on the current short rate. This means that it is enough to match the mean and the variance of this conditional distribution at every step of the tree to ensure that fixing the maturity of the instrument the process represented by the tree matches the short rate SDE (stochastic differential equation).

Example using the Merton (Ho-Lee) Model (Example taken from Tuckman page 227).

Remember that in the Merton (Ho-Lee) Model we have

$$dr = \alpha dt + \sigma d\hat{B}_t \tag{68}$$

Here the Euler discretization implies that

$$r_{t+\Delta} = r_t + \alpha\Delta + \sigma \times N(0, \Delta) \tag{69}$$

where N is a normal shock with zero mean and $\Delta$ variance.

Notice that the above implies

$$E_t(r_{t+\Delta}) = r_t + \alpha\Delta \tag{70}$$

and

$$Var_t(r_{t+\Delta}) = \sigma^2\Delta \tag{71}$$

Now consider the following discretization Notice that if we assign 0.5 probability for a rate hike or a fall we match the conditional mean and variance of the short rate at each step. We can take the numerical integration in Equation 1 using the tree. In particular, given any subtree we can compute

| | date 1 | date 2 |
|---|---|---|
| | | $r_t + 2\alpha\Delta + 2\sigma\sqrt{\Delta}$ |
| | $r_t + \alpha\Delta + \sigma\sqrt{\Delta}$ | |
| $r_t$ | | $r_t + 2\alpha\Delta$ |
| | $r_t + \alpha\Delta - \sigma\sqrt{\Delta}$ | |
| | | $r_t + 2\alpha\Delta - 2\sigma\sqrt{\Delta}$ |

Table 4. The two period tree for the Merton (Ho-Lee) Model

| | |
|---|---|
| | $B_{up,t+\Delta}$ |
| $r_t$ | |
| | $B_{down,t+\Delta}$ |

Table 5. Associating a $B$ value for every $r$

$$B_{state(upordown),t} = exp(-\Delta r_t) \times 0.5 \times (B_{up,t+\Delta} + B_{down,t+\Delta})$$

where $B_{state(upordown),T} = exp(-\Delta r_T)$. So we can work the tree backwards.

Lets price a 2 year bond using the Merton(Ho-Lee) tree. We set $\alpha = 0.01, \sigma = 0.03$ and $r_t = 0.05$. holeemertontree.m does the tree based pricing.

Lets also implement a tree for the Continuous-time Ho-Lee model (see Tuckman page 228). The difference here is that the drift is allowed to be time dependent. Again we have a recombining tree since the upchange is the same size as the downchange. Let's focus on drawing a risk-neutral tree out to five years. Remember that line 35 of contholee.m gives the $\alpha$ function out to thirty years.

The stochastic differential equation we have is

$$dr_t = \alpha(t)dt + \sigma\hat{B}_t \tag{72}$$

. The euler scheme for discretizing implies that

$$r_{t+\Delta} - r_t = \alpha(t)\Delta + \sigma\sqrt{\Delta}N(0,1). \tag{73}$$

Based on above the first two dates of the tree will look like;

| | date 1 | date 2 |
|---|---|---|
| | | $r_t + (\alpha_t + \alpha_{t+\Delta})\Delta + 2\sigma\sqrt{\Delta}$ |
| | $r_t + \alpha_t\Delta + \sigma\sqrt{\Delta}$ | |
| $r_t$ | | $r_t + (\alpha_t + \alpha_{t+\Delta})\Delta$ |
| | $r_t + \alpha_t\Delta - \sigma\sqrt{\Delta}$ | |
| | | $r_t + (\alpha_t + \alpha_{t+\Delta})\Delta - 2\sigma\sqrt{\Delta}$ |

Table 6. The two period tree for the (Continuous Ho-Lee) Model

contholeetree.m does the tree based pricing.

**Illustrating The Equivalence Between Simulation, Analytic Expectation and Tree Approaches**

We can consider the below example to illustrate the equivalence of Simulation, Analytic Expectation and Tree approaches. Consider a term structure model with only one parameter $\sigma$

$$dr_t = \sigma d\hat{B}_t \tag{74}$$

Lets call the above specification the simplified Merton model.

This really is the Merton (Ho-Lee) model with $\alpha = 0$. Therefore the analytic bond prices are given by setting $\alpha = 0$ in equation 45.

In particular,

$$\Lambda_{t,T} = E_t^Q\left[exp(\int_t^T -r_u du)\right] = exp(-(T-t)r_t + 1/2 \times \sigma^2 \times (T-t)^2/3) \tag{75}$$

simplifiedmerton.m finds the price of one to five year maturity bonds with $r_t = 10\%$ and $\sigma = 3\%$.

The Euler-discretized stochastic differential equation is

$$r_{t+\Delta} = r_t + \sigma\sqrt{\Delta}N(0,1) \tag{76}$$

We can simulate the above forward and perform simulation based pricing. simsimplifiedmerton.m does the simulation based pricing.

Finally we can solve for bond prices using a tree. This is the same tree as the Merton (Ho-Lee) but without any $\alpha$

| | date 1 | date 2 |
|---|---|---|
| | | $r_t + 2\sigma\sqrt{\Delta}$ |
| | $r_t + \sigma\sqrt{\Delta}$ | |
| $r_t$ | | $r_t$ |
| | $r_t - \sigma\sqrt{\Delta}$ | |
| | | $r_t - 2\sigma\sqrt{\Delta}$ |

Table 7. The two period tree for the simplified Merton Model

simplifiedmertontree.m does the tree based pricing.

**Modification of simplifiedmertontree.m Eliminating the Keeping of the Entire Tree in the Memory.**

simplifiedmertontreemem.m does the modification.

Introduction to Solving Term Structure Models using Differential Equations (Duffie page 136-137).

If the drift $\mu(r_t, t)$ and the instantaneous variance $\sigma^2(r_t, t)$ are affine in $r_t$ in that $\mu(r_t, t) = \alpha_1(t) + \alpha_2(t)r_t$ and $\sigma^2(r_t, t) = \beta_1(t) + \beta_2(t)r_t$, the bond price that solves the pde specification 7 and 8 can be shown to be of the form $f(x, t) = exp(a(T - t) + b(T - t)x)$ where $b$ and $a$ solve the following ODEs.

$$b'(\tau) = \alpha_2(\tau)b(\tau) + 1/2\beta_2(\tau)b^2(\tau) - 1; b(0) = 0 \tag{77}$$

and

$$a'(\tau) = \alpha_1 b(\tau) + 1/2\beta_1 b^2(\tau); a(0) = 0 \tag{78}$$

Such a term structure model is called an Affine Single factor term structure model. Thus technicalities aside, $\mu$ and $\sigma^2$ are affine in $r$ only if the yields are affine in $r$ (i.e. prices are exponential affine). Furthermore the above system equations are called the Ricatti equation. The Ricatti equation is easy to solve using discretization methods like Runge-Kutta. So, if we face a specification of the short rate process where the $\mu(r_t, t)$ and the instantaneous variance $\sigma^2(r_t, t)$ are affine in $r_t$ instead of dealing with the harder pde in equations 7 and 8 we can deal with the easier to solve above ODEs.

It is easy to see that CIR, Vasicek, Merton(Ho-Lee), Hull and White, Continuous-time Ho-Lee are all affine models.

**Solving the Vasicek model by PDE Methods.**

Notice that in the Vasicek model;

$\alpha_1 = A\bar{x}$ and $\alpha_2 = -A$ and $\beta_1 = \sigma^2$ and $\beta_2 = 0$.

So, we have the below ODE system for the model.

$$b'(\tau) = -Ab(\tau) - 1; b(0) = 0 \tag{79}$$

and

$$a'(\tau) = A\bar{x}b(\tau) + 1/2\sigma^2 b^2(\tau); a(0) = 0 \tag{80}$$

Vasicekbyode.m performs the ode based pricing. ode45 is the ode solver you should probably try first.

**Solving the Continuous Ho-Lee Model by PDE methods**

Notice that in this model, $\alpha_1 = \alpha(t)$, $\alpha_2 = 0$, $\beta_1 = \sigma^2$ and $\beta_2 = 0$.

So the ODE system here is

$$b'(\tau) = -1; b(0) = 0 \tag{81}$$

$$a'(\tau) = \alpha(t)b(\tau) + 0.5\sigma^2 b^2(\tau); a(0) = 0; \tag{82}$$

We can immediately solve for $b(\tau)$ as $b(\tau) = -\tau$

Therefore the only ODE we have is

$$a'(\tau) = -\alpha(\tau)\tau + 0.5\sigma^2\tau^2; a(0) = 0; \tag{83}$$

contholeebyode calculates the bond prices given the $\alpha(t)$ function that prices the initial term structure exactly (this $\alpha(t)$ function is computed by exploit equation 67.

**Solving the CIR model by PDE methods**

In the model $\alpha_1 = A\bar{x}$, $\alpha_2 = -A$, $\beta_1 = 0$ and $\beta_2 = C^2$. So the ODE system becomes;

$$b'(\tau) = -Ab(\tau) + 1/2C^2 b^2(\tau) - 1; b(0) = 0 \tag{84}$$

and

$$a'(\tau) = A\bar{x}b(\tau); a(0) = 0 \tag{85}$$

cirbyode.m calculates bondprices given $A$,$\bar{x}$ and $C$.

**The Black-Karasinski Model**

In this model the short rate dynamics under the risk neutral measure is given as;

$$dr_t = \alpha_2(t)r_t + \alpha_3(t)r_t log(r_t)dt + \beta_2(t)r_t d\hat{B}_t \tag{86}$$

Notice that the Black-Karasinski model is not affine. So we can not collapse the associated PDE for bond prices to a set of ODEs. As a first pass, lets price some bonds using the simulation approach. simblackkarasinski.m does this. Notice that the program turns off t-dependence of $\alpha_2$, $\alpha_3$ and $\beta_2$.

**The Black-Derman-Toy (BDT) Model**

The model is written as

$$r_t = U_t exp(\gamma_t B_t) \tag{87}$$

where $U_t = U(t)$ and $B_t$ is is a brownian motion in measure Q.

Using Ito's lemma (see Duffie page 85); we can write

$$dr_t = \left[U_t \gamma_t exp(\gamma_t B_t)0 + U_t' exp(\gamma_t B_t) + U_t B_t \gamma_t' exp(\gamma_t B_t) + u_t \gamma_t^2 exp(\gamma_t B_t)\right] dt + U_t \gamma_t exp(\gamma_t B_t) dB_t \tag{88}$$

simplifying and substituting $r_t$ for $U_t exp(\gamma_t B_t)$ we obtain

$$dr_t = r_t \left[U_t'/U_t + B_t \gamma_t' + \gamma_t^2\right] dt + r_t \gamma_t dB_t \tag{89}$$

Apply Ito's lemma again to obtain

$$dlogr_t = \left[U_t'/U_t + B_t \gamma_t' + 1/2\gamma_t^2\right] dt + \gamma_t dB_t \tag{90}$$

so defining $g(t) = [U_t'/U_t + B_t \gamma_t' + 1/2\gamma_t^2]$, the short rate SDE in the Black-Derman-Toy model can be written as

$$dlogr_t = g_t dt + \gamma_t dB_t \tag{91}$$

Lets take $\gamma_t$ as a constant $\gamma$ then, we obtain the below simplified Black-Derman-Toy model. Notice that, this version of the black-derman-toy model is the same as the Continiuos Ho-Lee model but the SDE is defined for the $logr_t$ rather than $r_t$.

$$dlogr_t = g_t dt + \gamma B_t \tag{92}$$

There is no analytic result for bond prices for the BDT model but we can calibrate the model using simulation or the tree methods. The below paragraphs implement the calibration exercise using simulation and tree methods and the point is calibration by tree is a more reliable way to go. The noise in finite simulation interferes with the updates of the search procedure in a bad way so as to make the calibration by simulation sensitive to the initial choice of parameters (parameters that define, g(t)).

**Calibrating the Model by Simulation**

Notice that

$$logr_{t+\Delta} = logr_t + g(t) * \Delta + \gamma\sqrt{\Delta}N(0,1) \tag{93}$$

We simulate log(r), but each each step we exponentiate the log(r)s to r's.

The g(t) is calibrated as step function with n breakpoints $t_1, t_2, ...t_n$ to match n initial yields and coincide with the data bond maturities. $g(0)$ is set to $g(t_1)$. The optimizer iterates on $g(t_1), g(t_2), ...g(t_n)$ to price the n bonds exactly.

simbdt.m does calibrate the model by simulation to match 1 month and 3 month bond prices. Notice that the initial values provided do a good job in making the optimize converge. If you start from another initial set you do not end up matching bond prices well.

**Calibrating the Model by Tree**

g(t) can be defined again as a step function, for any g(t) we can construct the tree

| | date 1 | date 2 |
|---|---|---|
| | | $log(r_t) + (g_t + g_{t+\Delta})\Delta + 2\gamma\sqrt{\Delta}$ |
| | $log(r_t) + g_t\Delta + \gamma\sqrt{\Delta}$ | |
| $log(r_t)$ | | $log(r_t) + (g_t + g_{t+\Delta})\Delta$ |
| | $log(r_t) + g_t\Delta - \gamma\sqrt{\Delta}$ | |
| | | $log(r_t) + (g_t + g_{t+\Delta})\Delta - 2\gamma\sqrt{\Delta}$ |

Table 8. The two period tree for the BDT model

Notice that we first construct the log(r) tree, then exponentiate to obtain the r tree then price bonds. Notice that the optimizer quickly and accurately goes from [1, 1] to [1.1218, 1.6016] as $g(t_1)$ and $g_(t_2)$. The optimizer iterates better on the tree based approach compared to the simulation based approach. bdttree.m calibrates the BDT model using the tree.

**Pricing Coupon Bearing Bonds**

Treasury securities are issued as either discount or coupon securities. Discount securities pay a fixed amount at maturity, called face value or par value. Coupon securities are issued with a stated rate of interest, pay interest every six months and pay the face value at maturity. Older issues of a given maturity are called off-the run. Trading volume is much higher for on-the-run then off-the-run.

| Issue | Auction Frequency | Type |
|---|---|---|
| 4-week bill | weekly | discount |
| 13-week bill | weekly | discount |
| 26-week bill | weekly | discount |
| 2-year note | monthly | coupon |
| 3-year note | quarterly | coupon |
| 5-year note | monthly | coupon |
| 10-year note | quarterly | coupon |

Table 9. Merged Exhibit 10-1 and 10-3 in Fabozzi

**Treasury Bond Cash Flows and Quotations**

The cash flows from Treasury bonds are completely defined by face value , coupon rate and maturity date.

Consider a purchasing a Treasury bond on 8/15/98 with 10000 face value, a coupon rate of 5 1/4/% and a maturity of August 15, 2003.

The cash flows are given by the table below.

| Day | Cash Flow |
|---|---|
| 8/15/98 | -price |
| 2/15/99 | 262.50 |
| 8/15/99 | 262.50 |
| 2/15/00 | 262.50 |
| 8/15/00 | 262.50 |
| 2/15/01 | 262.50 |
| 8/15/01 | 262.50 |
| 2/15/02 | 262.50 |
| 8/15/02 | 262.50 |
| 2/15/03 | 262.50 |
| 8/15/03 | 262.50+10000 |

Table 10. Figure 1-1 in Tuckman. $262.50 = 0.0525 \times 10000/2$

The below table gives the bond prices on Feb 15, 2001 (Settlement day)

| Coupon | Maturity | Price |
|---|---|---|
| 7.875% | 8/15/01 | 101-12 3/4 |
| 14.250% | 2/15/02 | 108-31+ |
| 6.375% | 8/15/02 | 102-5 |
| 6.250% | 2/15/03 | 102-18 1/8 |
| 5.250% | 8/15/03 | 100-27 |

Table 11. Table 1-1 in Tuckman

By convention price should be interpreted as percent of face value whenever a dollar or other currency symbol does not appear. Note that number after the hyphens (-) denoted 32nds, often called ticks. The sign, +, means a half tick.

For the 7 7/8s of Aug, 2001, the price for a \$10000 face value is \$10000(101+(12+3/4)/32)/100=\$10139.84.

For the 14 1/4s of Feb, 2001, the price for a \$10000 face value is \$10000(108+(31+1/2)/32)/100=\$10898.44.

Lets price the 14 1/4s using the Vasicek model. couponvasicek.m does the pricing. Here the matlab built in function cfamounts.m comes handy. Notice that the first entry of the cashflow output is the accrued interest.

Instead of calibrating the term-structure model to price the zero curve exactly, we can directly calibrate it to price a set of bond price quotes. calibratecouponvasicek.m does the calibration to the first four (A,$\bar{X}, \sigma, r_t$ being the four parameters of the square system) bonds in Table 1-1 in Tuckman provided above.

**Pricing a European Bond Option using the Black Derman Toy Tree**

Consider a European Bond Call option on a now (time t) 6-month maturity T-bill. The expiration is 3 months from now and the strike K. The face value of the Bill is \$1000. The call payoff at $t + 3/12$ is $max(P_{t+3/12,t+6/12} - K, 0)$. where $P_{t,\tau}$ denotes the time $t$ price of a bond maturing at time $t+\tau$. Below, $t+\tau_1$ is the expiry, K is the strike $t+\tau_2$ is the bill maturity at time t, F is the bond face value.

$$O_{t,t+\tau_1,t+\tau_2,K,F} = E_t\left[exp(-\int_t^{t+\tau_1} r_u du) \times max(F \times P_{t+\tau_1,t+\tau_2} - K, 0)\right] \tag{94}$$

The strategy is to obtain a calibrated BDT tree up to 6 months. Then replace bond pay-off's at half-way through the tree with the associated call pay offs; C=max(B-K,0).

BDTtreeeurooption.m does the pricing of a european call on a T-bill using a calibrated BDT tree.

**Pricing a European Bond Option using the Merton(Ho-Lee) tree.**

holeemertontreeeurooption.m does the pricing of a european call on a T-bill using a Merton(Ho-Lee) tree.

**Using the Calibrated Vasicek Model with simulations to Price Bond options on Coupon Bearing Bonds**

Remember that vasicekopt.m calibrated the $A, \bar{x}, \sigma$ to do a good job in pricing the historical zero curves. Lets use the calibration to price a European option on a coupon bearing bond.

Let the expiration of the option be $\tau_1$ and let the cash flows of the bond be indexed $\tau_2$, ...$\tau_n$, where $\tau 1 < \tau_2 < \ldots < \tau_n$. The value of the bond at expiry, $V(r_{\tau_1}, \tau_1, \tau_2, \tau_3, \ldots, \tau_n)$ is $Vasicek(r_{\tau_1}, \tau_2 - \tau_1 \times CF_{\tau_2}) + Vasicek(r_{\tau_1}, \tau_3 - \tau_1 \times CF_{\tau_3}) + \ldots + Vasicek(r_{\tau_1}, \tau_n - \tau_1 \times CF_{\tau_n})$, where $Vasicek(r_t, \tau, CF)$ is the time t dollar value of a zero coupon bond in the Vasicek model where the short rate is $r_t$, the maturity is $\tau$ and the face value is CF. We ignore accured interest(This would reduce the bond value). Once the value of the bond is known the value of the option is also known. Say we are interested in a European call with strike K and expiry $\tau_1$. The call value at $\tau_1$is $max(V(r_{\tau_1}, \tau_1, \tau_2, \tau_3, \ldots, \tau_n) - K, 0)$. The price of the call at time $t < \tau_1$ is $E\left[exp(-\int_t^{\tau_1} r_u du) max(V(r_{\tau_1}, \tau_1, \tau_2, \tau_3, \ldots, \tau_n) - K, 0)\right]$. This expectation could be taken numerically.

vasicekeurobondoptionsim.m does the simulation based pricing.

**Using the CIR Model with simulations to Price Bond options on Coupon Bearing Bonds**

cireurobondoptionsim.m does the simulation based pricing.

**Valuing a Caplet**

A caplet is an instrument which pays off if the yield on a benchmark bond is higher than a predetermined number (cap rate $\hat{R}$). It can be idealized to as a contract initiated at time t,that pays the purchaser $NP \times max(y_{\tau_1,T} - \hat{R}, 0)$ where $t < \tau_1 < T$ and $y_{\tau_1,T}$ denotes the yield at time $\tau_1$ of the benchmark bond that matures at time $T$. NP is the notional principal. The benchmark rate is often set as a function of the LIBOR rate. There are various LIBOR rates most liquid ones are 1-month, 3-month and 12-month. We will need to fit a term structure model to the LIBOR market. We will perform this calibration using the EuroDollar Futures market more carefully later. Lets assume for now the Benchmark rate is the yield on a three-month t-bill. Lets consider a caplet with the payoff scheduled for one month from now. Further lets fix the notional principal at a \$1000. Lets price this caplet by the calibrated CIR model.

The price is

$$C_t = E_t^Q \left[ exp(-\int_t^{t+1/12} r_u du) max(y_{t+1/12,t+4/12} - \hat{R}, 0) NP \right] \tag{95}$$

Notice that in the CIR model the yields are analytic. In particular,

$$y_{t,s} = (-log(H_1(s-t) + H_2(s-t)r_t)/(s-t) \tag{96}$$

where $H_1$ and $H_2$ are defined in equations 13 and 14.

circaplet.m implements the pricing through simulations.

A floorlet can be defined similarly as a contract that pays if the a yield on a benchmark bond is below a certain rate (floor rate).

For a floor where the floor rate is $\hat{R}$, the benchmark maturity is $T - \tau_1$ and payoff day $\tau_1$ we can write the time t value as belows.

$$F_t = E_t^Q \left[ exp(-\int_t^{t+\tau_1} r_u du) max(\hat{R} - y_{\tau_1,T}, 0) NP \right] \tag{97}$$

circaplet.m has a final line that computes the floorlet price.

**Valuing a caplet and a floorlet using the Continuous Ho-Lee model rather than the Vasicek Model.**

contholee.m implements the pricing through simulations.

**The Multifactor CIR model** (Duffie page 144).

A more often used multifactor model is the multifactor version of the CIR model. Lets consider a d-factor CIR model. The factors $X_i$, i=1,2, ...,d evolve according to;

$$dX_{it} = A_i(\bar{x}_i - X_i t)dt + C_i \sqrt{X_i t} d\hat{B}_t^i \tag{98}$$

; where $X_{i,0} > 0$ and $A_i, \bar{x}_i, C_i$ are positive constants, analogous to $A, \bar{x}$ and C in the single factor CIR model. The short rate is modelled as the sum of all factors; $r_t = \sum_{i=1}^{d} X_{it}$.

It can be showed that $\hat{B}^i$ and $\hat{B}^j$ are independent under Q means that the factor processes are independent as well.

So, we start with;

$$\Lambda_{t,s} = E_t^Q \left[ exp(-\int_t^s r_u du) \right] \tag{99}$$

$$= E_t^Q \left[ exp(-\int_t^s (\sum_{i=1}^{d} X_{iu}) du) \right] \tag{100}$$

$$= E_t^Q \left[ exp(-\sum_{i=1}^{d} \int_t^s X_{iu} du) \right] \tag{101}$$

$$= E_t^Q \left[ \Pi_{i=1}^d exp(-\int_t^s X_{iu} du) \right] \tag{102}$$

Invoking independence of the factor processes;

$$= \Pi_{i=1}^d E_t^Q \left[ exp(-\int_t^s X_{iu} du) \right] \tag{103}$$

$$= H_{1i}(s-t) exp(-H_{2i}(s-t) X_{it}) \tag{104}$$

$$\Lambda_{t,s} = exp(-h(s-t) - H_2(s-t).X_t) \tag{105}$$

where $h(s-t) = -\sum_{i=1}^d log(H_{1i}(s-t))$ and $H_2(t) = [H_{21}(t), H_{22}(t), \ldots, H_{2d}(t)]$ and . stands for the dot product operator (for n by 1 vector x and y x.y is defined to be $x_1 * y_1 + \ldots x_n * y_n$. Functions $H_{1i}$ and $H_{2i}$ are defined analogously to the one factor CIR case but with factor dependent constants $A_i, \bar{x_i}$ and $C_i$.

**Interest Rate Swaps**

From non-existence in 1980, swaps have grown into a very large and liquid market where participants manage their interest rate risk. Consider a fixed for floating swap initiated Nov 26, 2001 where Party A agrees to pay 5.688% on 100Million to Party B every 6 months form 10 years where party B agrees to pay the three-month LIBOR for 10 years. The first few cash flows are as belows.

| Date | 3m LIBOR | Date | Actual Days | Floating Receipt | 30/360 days | fixed |
|---|---|---|---|---|---|---|
| 11/26/01 | 2.156% | 11/28/01 | — | — | — | — |
| 02/26/02 | 2.000% | 02/28/02 | 92 | 550883 | 90 | — |
| 05/26/02 | 1.900% | 05/28/02 | 89 | 494444 | 90 | 2844000 |
| 8/26/02 | 2.000% | 08/28/02 | 92 | 485556 | 90 | — |
| 11/26/02 | 2.100% | 11/29/02 | 93 | 516667 | 91 | 2859800 |

Table 12. Partial Table 18-1 in Tuckman

Notice that 100000000 × 2% × 89/360 is 494444, since the floating rate cash flows are determined using "actual/360" convention.

The fixed rate cash flows are determined using the 30/360 convention, so $100000000 \times 5.688\%(90+91)/360$ is 2859800.

Lets use the vasicek model to price this swap for the party paying the floating leg. vasicekswap.m does the pricing.

An important feature of the floating leg is that the libor rate 3 months ago determines today's pay. vasicekswap incorporates this feature into the pricing.

**Calculating the par-swap rate**

This is the fixed rate that makes the value of the swap zero for both parties so they can both enter the contract for free.

Lets calculate the value of the swap contract using the BDT tree then iterate on the fixed rate to make it the swap par. bdttreefloatingleg.m and bdttreeparswapcaller.m perform these.

Notice that BDT tree implementation assumes that the libor rate of today determines the floating pay today not the libor of 3 months ago.

**Swaptions**

A swaption is a contract which allows the purchaser to enter a swap contract at a future date. In vasicekswap.m the swap value to the floatingpayer is a function of the short rate. Once we have the functional form as a function of the short rate, we can price a european swaption with payoff $V_{\mathcal{T}} = max(valueofswapforfloatingpayer(r_{\mathcal{T}}), 0)$ as $E\left[exp(-\int_t^{\mathcal{T}})max(valueofswapforfloatingpayer(r_{\mathcal{T}}), 0)\right]$ by means of simulation.

vasicekswaption.m first obtains a representation for the functional form for the $r_{\mathcal{T}}$ and the $V_{\mathcal{T}}$ for a range of short rate values then performs a simulation-based pricing of the swaption.

**The Heath-Jarrow-Morton Model of Forward Rates** (Duffie page 149, Glasserman page 153).

Thus far in modelling the term structure we always started with a process for the short rate under the risk neutral measure and taking the expectation in equation 1. Instead of starting from a model of the short rate, HJM start from a model of forward rates.

We have defined forward rates at equation 63. Recall that the notation forward(t,u) (f(t,u))means the logarithmic rate of return that can be locked in at time t for investing at a future time u, where the duration of the investment is infinitesimal.

Notice that the current, time t, zero curve, i.e. the schedule of $\Lambda_t, s$ as a function of $s$ allows to compute all forward rates f(t,s) and knowledge of the forward curve allows to compute the zero curve. In particular, the definition of the forward rate immediately means that,

$$\Lambda_{t,s} = exp(-\int_t^s f(t,u)du) \tag{106}$$

.

The HJM model of forward rates start with the evolution of the forward rate through time.

$$f(t,s) = f(0,s) + \int_0^t \mu(u,s)du + \int_0^t \sigma(u,s)d\hat{B}_u \tag{107}$$

for 0¡t¡s.

Or in differential form

$$df_t = \mu(t,s) + \sigma(t,s)d\hat{B}_t \tag{108}$$

Notice that for any $t$, $f(t,t) = r_t$, the short rate.

It turns out that for equation 107 to be consistent with lack of arbitrage it has to be that

$$\mu(t,s) = \sigma(t,s)\int_s^t \sigma(t,u)^T du \tag{109}$$

, where $T$ denotes transposition.

Therefore,

$$df_t = \sigma(t,s)\int_s^t \sigma(t,u)^T du + \sigma(t,s)d\hat{B}_t \tag{110}$$

For a proof see Duffie page 151-153.

It is useful to write the forward rate dynamics in summation form as

$$d_f(t,T) = \sum_{j=1}^{d}\left(\sigma_j(t,T)\int_t^T \sigma_j(t,u)du\right)dt + \sum_{j=1}^{d}\sigma_j(t,T)d\hat{B}_j(t) \tag{111}$$

Notice that we are allowing $d\hat{B}^t$ to be $d$ dimensional where d can be greater than 1. $\mu(u,s)$ is one dimensional.

Notice

We can use $r_t = f(t,t)$ and get a process for the short rate. In particular,

$$r_t = f(0,t) + \int_0^t \sigma(v,t)\int_v^t \sigma(v,u)^T dudv + \int_0^t \sigma(v,t)d\hat{B}_v \tag{112}$$

Given equation 112 we can take the expectation in equation 1. Notice that this process will not be markovian which will make simulating the simulation procedure differ a bit from earlier examples.

The HJM model often collapses to some of the short rate models we have studied earlier.

For example set $\sigma(t,s) = \sigma$ (the constant instantaneous forward volatility HJM model), then

$$df(t,s) = \sigma^2(s-t)dt + \sigma d\hat{B}_t \tag{113}$$

$$f(t,s) = f(0,s) + \sigma^2(st - t^2/2) + \sigma\hat{B}_t \tag{114}$$

Using $r_t = f(t,t)$, then we obtain,

$$r_t = f(0,t) + 1/2\sigma^2 t^2 + \sigma\hat{B}_t \tag{115}$$

Applying ito's lemma we obtain,

$$dr_t = (f'(0,t) + \sigma^2 t)dt + \sigma d\hat{B}_t \tag{116}$$

, where f'(0,t) is the first derivative with respect to t of the time-0 forward curve. The initial condition is $r_0 = f(0,0)$. In this specification, the HJM coincides with the Continuous Ho-Lee Model.

It can be shown that the Vasicek and CIR are also special cases of the HJM framework with $\mu$ and $\sigma$ specified appropriately(see page 153 and 157 of Baxter and Rennie.

In equation 110 the drift is determined once the $\sigma$ function is specified. This contrasts with the short rate models where the drift and the volatility could be specified independently. In fact choosing the parameters of the drift was important in matching the initial yield curve. In the HJM model once the f(0,t) function is chosen as the negative of the first derivative of the bond prices with respect to maturity, the model automatically prices the initial term structure exactly. The calibration exercise in the HJM model focusses on the $\sigma$ function which is used to get to model match a certain set of *interest rate derivative* prices.

hjm.m implements the hjm specification.

**Swap Contracts where the Floating Pay is dependent on the Simple Compounded Yield rather than the Continuously Compounded Yield**

An overwhelming majority of fixed for floating swaps are simply compounded rather than continuously compounded. When the swap is a simply compounded swap valuing the floating leg becomes very easy.

In a simply compounded swap, the time $t_2$ floating pay is

$$(NP \times [1/P_{t_1,t_2} - 1]/(t_2 - t_1)) \times (t_2 - t_1) \tag{117}$$

(for a three month libor swap $t_2 - t_1 = 0.25$.). $[1/P_{t_1,t_2} - 1]/(t_2 - t_1)$ is called the simply compounded spot rate.

In a continuously compounded swap, the time $t_2$ floating pay is

$$NP \times [-log(P(t_1, t_2))/(t_2 - t_1)] \times (t_2 - t_1) \tag{118}$$

(see for example vasicekswap.m). $[-log(P(t_1, t_2))/(t_2 - t_1)]$ is called the continuously compunded spot rate.

Consider a strategy where the investor buys at time 0, a $t_1$ maturity bond and shorting a $t_2$ maturity bond.

The cash flow to the investor is $-NP \times P(0, t_1) + NP \times P(0, t_2)$, (notice that this is negative). The time 1 value of the long-short portfolio is $+NP - NP \times P(t_1, t_2)$ (a positive). This can then be invested on a one period bond then yielding $(NP - NP \times P(t_1, t_2))/(P(t_1, t_2))$ at time t, certainly. So regardless of the circumstance a portfolio put together by a cost of $NP \times P(0, t_1) - NP \times P(0, t_2)$ (notice that this is positive) yields a dollar amount equal to the floating pay, so this quantity should be the value of the floating component. So the value of the first floating leg is $NP \times (P(0, t_1) - P(0, t_2))$ .

Recall that the fixed payment at time $t_2$ is $NP \times X \times t_2 - t_1$. So the present value of this time $t_2$ cash flow is $NP \times X \times t_2 - t_1$, where X is the fixed rate.

Lets call the $t_{i+1} - t_i = \tau_i$ (the difference between the payment date and the reset date for period i).

Then the present value of the fixed payments is;

$$PV(fixed) = \sum_{i=1}^{n} NP \times X \times \tau_i P(0, t_{i+1}). \tag{119}$$

We have set $t_1$ as the entry date and therefore $t_2$ as the first date at which money is exchanged. There are a total of $n$ exchanges.

$$PV(floating) = \sum_{i=1}^{n} NP \times (P(0,t_i) - P(0,t_i+1)). \tag{120}$$

Let's introduce definition for simply compounded forward rate as a measure of the rate that can be locked in at time $t_1$ for investing-borrowing between $t_i$ and $t_{i+1}$.

$$F_i \equiv F(t_1,t_i,t_{i+1}) = (P(t_1,t_i)/P(t_1,t_{i+1}) - 1)/(t_{i+1} - t_i) \tag{121}$$

.

Equating the PV(fixed) and PV(floating) and putting X to the left hand side we can state the par-swap rate.

$$X = \frac{\sum_{i=1}^{n} NP \times F_i \times \tau_i \times P(0,t_{i+1})}{\sum_{i=1}^{n} NP \times \tau_i \times P(0,t_{i+1})} \tag{122}$$

In some non-plain vanilla swaps the NP is made a function of time.

Then

$$X = \frac{\sum_{i=1}^{n} NP_i \times F_i \times \tau_i \times P(0,t_{i+1})}{\sum_{i=1}^{n} NP_i \times \tau_i \times P(0,t_{i+1})} \tag{123}$$

Notice we can rewrite, the par swap rate as a weighted average of the simple compounded forward rates:

$$X = \Sigma F_i \times w_i \tag{124}$$

, where

$$w_i = \frac{NP_i \tau_i P(0,t_{i+1})}{\sum_{i=1}^{n} NP_i \tau_i P(0,t_{i+1})} \tag{125}$$

.

**par-swap curve** We can plot X as function of maturity of the last payment $t_{n+1}$. This curve is called the (for any day, we plot the weighted average forward rates till date day, where the weighted average is taken as above, see Tuckman figure 18.2).

**Swap Spreads**

Swap spreads are simply the difference between (par) swap rates for a maturity and government bond yields of the same maturity. It is hard to find a T-bond maturing exactly in say 10 years so we can use any on-the-run (liquid) T-bond yield fro 10 year maturity. Swap spreads are watched closely by the fixed-income industry. It is hard to put a precise interpretation to swap spreads. One interpretation is that a larger spread means there is a larger default premium in the market. A larger spread means that credit is less available. For example, in Fall of 1998, highly volatile swap spreads led the US Fed to cut short-term interest rates by 75 basis points. One use of a swap is for a corporation to hedge against adverse moves in rates for future issuance of debt. If it has decided to issue debt, they would enter into a fixed-payer swap where they receive money on the swap if rates go up, but lose since now they have to offer a higher coupon rate.

**Midterm Examination: Due Wed 13th**

Please comment your code so that it is readible. For each question write the theoretical setup separately (much like the class notes). Make sure the comments in the code correspond to your theoretical analysis.

1) Consider a bond option of european type on a zero coupon bill. Calibrate the Continuous-Ho Lee model to the last observation of the Fama-Bliss data set. Write a program that can handle puts, calls, and takes the key parameters as inputs.

2) Calibrate the BDT tree to the current term structure off Bloomberg. Write a program to price a caplet or a floorlet and takes the key parameters as inputs.

3) Write a program to compute the par swap rate for the current term structure off Bloomberg. Allow your code flexility to handle notional principles as function of the date, 6 month or a 3 month floating reset dates, 6 month or a 3 month fixed dates. Calculate the par swap rate. Provide a plot of the par-swap curve.

4) Write a program for the HJM specification with the volatility surface $\sigma_(t,s) = K \times \sqrt{(s-t)}$, where K is a constant. Price the current term structure exactly (from Bloomberg). Then price a caplet by simulations.

**Bill and Bond Futures**

The market futures price is such that entering a futures contract requires no initial cash flow. There are daily cash in or out flows and daily flow to the long (short) position is the (negative) change in the market futures price over the day till the expiry of the futures contract. At the expiry of the futures contract the futures price is the price of the underlying.

We are interested in how the futures price $\Phi_t$ moves through time. We know that $\Phi_T = W_T$ where, T is the expiration and $W_T$ is the underlying spot price at expiration. This is the terminal futures price. We know one more thing about the futures price: its dividend flow (daily cash in or outflows)has value zero.

Defining

$$Y_t = exp(-\int_0^t r_s ds) \tag{126}$$

.

Then we have

$$0 = E_t^Q(\int_t^T Y_s d\Phi_s) \tag{127}$$

Notice that it is easy to see that the above equation is the limit (as $\Delta$ goes to zero of),

$$0 = E_t^Q \sum_{i=0}^{T} \Delta Y_{t+i\times\Delta} \left[\Phi_{t+(i+1)\times\Delta} - \Phi_{t+i\times\Delta}\right] \tag{128}$$

It can be shown that equation 127 and the terminal condition $\phi_T = W_T$ imply that the futures price process is a martingale, therefore,

$$\phi_t = E_t^Q(W_T) \tag{129}$$

Equation 129 determines the continuously settled futures price. We simply take the expectation of the underlying's value under the risk neutral measure.

The most popular interest rate futures contract in the United States is the **3-month Eurodollar futures contract** traded on the Chicago Mercantile Exchange (CME). Eurodollar is a dollar deposited in a US or foreign bank outside the United States. The Eurodollar interest rate is the interest rate earned on Eurodollars deposited by one bank with another bank. It is essentially the same as the LIBOR (Hull, page 137). There-month Eurodollar futures contracts are futures contracts where the terminal cash flow is $100(1 - R_T$, where $R_T$ is the actual three-month eurodollar interest rate that day, $100(1 - R_T)$. $R_T$ is quarterly compounded $R_T = \left[(1/P_{T,T+0.25})^{0.25} - 1\right]/0.25$.

Lets calculate the futures price on a Futures expiring 6 months from now on a three month eurodollar. Lets use the BDT tree for this purpose. We extend the three up to 9 months, work back 3 months to find the bond price then, maturing three months from then. Then we simply take the expectation. Notice that if there are n nodes as of that date the probabilities of each node is simply $C(n, i)/2^n$ for $i = 0, 1, 2, \ldots, n$ where C is the combinatorial operator, i.e. C(N,K)=factorial(N)./(factorial(N-K)× factorial(K)). eurodollarfuturebdttree.m does the pricing. We can think of the nth step ahead distribution as normal with mean $n/2$ and variance $n \times 1/2 \times 1/2$. So we can weigh the ith node as, normpdf(i, $n/2, \sqrt{n \times 1/2 \times 1/2}$). Recall that normpdf.m is a matlab built in function that outputs the normal pdf value when the mean and the standard deviation are input. In the end, we need to sum the weights and divide each weight by the sum so that normalized weights sum to 1. Notice that the combinatorial algorithm will be exact but will be unstable when there is a large number of nodes. The normpdf approach is more stable yet approximate, though the approximation error goes to zero as the number of steps go to infinity fixing the maturity of the futures.

The euro dollar futures help investors lock in interest rates for the future investment since gains will be there day by day as the futures price pulls to the low spot.

The contracts have maturities up to 10 years Euro dollar futures contracts allow investors to essentially lock in an interest rate for a three month period 10 years from now.

**Hedging in a One factor world**

Think about a target derivative security (A) (or a bond) with price at time t , is $F(r_t, t)$. The derivative asset's sensitivity to the factor is $F_r(r_t, t)$, the first derivative with respect to r.

Consider another derivative security (B)(or a bond) with the price process $\Phi(r_t, t)$. The derivative asset's sensitivity to the factor is $\Phi(r_t, t)$, the first derivative with respect to r.

It turns out; if at time t, one longs $\Theta_t$ of the B-asset and invests an amount $F_r(r_t, t) - \Theta_t \times$

$\Phi(r_t, t)$ in the money market the one has a bundle that replicates the derivative value through time, where

$$\Theta_t = F_r(r_t, t)/\Phi_r(r_t, t) \tag{130}$$

.

It turns out the bundle is self financing (see Duffie page 141 for proofs).

Notice that the definition of $\Theta_t$ is very intuitive. If the value of the target derivative goes up 3 units when short rate changes 1 unit and the asset B only goes up by 2 units, then we need to have one and a half of asset B in the synthetic replicating portfolio.

**Hedging in a Multi-factor world**

Consider a derivative security whose price is can be written as $F(r_t, x_t, t)$, where $x_t$ is a second factor effecting the derivative price. Consider two instrument securities $\phi^1(r_t, x_t, t)$ and $\phi^2(r_t, x_t, t)$.

Let $\Theta_1(t)$ and $\Theta_2(t)$ be the hedge ratios.

Then we need

$$\Theta_1(t) \times \phi_1^1(r_t, x_t, t) + \Theta_2(t) \times \phi_1^2(r_t, x_t, t) = F_1(r_t, x_t, t) \tag{131}$$

and

$$\Theta_1(t) \times \phi_2^1(r_t, x_t, t) + \Theta_2(t) \times \phi_2^2(r_t, x_t, t) = F_2(r_t, x_t, t) \tag{132}$$

The two equations above uniquely pin down $\Theta_1(t)$ and $\Theta_2(t)$. The amount $F(r_t, x_t, t) - \Theta_1(t) \times \phi_1(r_t, x_t, t) - \Theta_2(t) \times \phi_2(r_t, x_t, t)$ should be invested in the money market. Of course one goes long $\Theta_1$ units of $\phi_1$ and $\Theta_2$ units of $\phi_2$.

**Hedging a Bond Option with the underlying bond in the BDT set-up**

Consider the same European Bond Call option on a 3month bill, expiring 3 months from now with strike K. We have BDTtreeeurooption.m did the pricing of this European call. Lets hedge his call with the underlying bond which is 6 months to maturity now. What is the constituents of the replicating portfolio? BDTtreeeurooptionhedge.m figures out the components of the hedging portfolio. The code shows that for the call option the replicating portfolio is long in the underlying bond and short in the money market. If one wants to cancel the exposure to the changes in call value, one would short the underlying bond and go long in the money market.

**Mortgage-Backed Securities**

Ownership of an MBS entitles one to the cash flows of a mortgage pool (a collection of individual mortgages).

Important terms

1) Prepayment models: Models we use to estimate future cash flows of a bond rather than scheduled cash flows

2) The PSA model (The Public Securities Association) The benchmark model for estimating prepayments.

3) CPR (Conditional Prepayment Rate). CPR is the proportion of the remaining mortgage balance thats is prepaid each month both quoted annualized.

4) SMM (Single-Monthly Mortality Rate): $(1-(1-CPR))^1/12$.

Estimated monthly prepayment is

Montly payment= SMM× [Beg. of Month balance-Sched. prin. for month].

In the PSA (100%) model, CPR depends on the maturity of the mortgages. In particular,

$$t <= 30 \quad , \quad CPR = 0.06 \times t/30 \tag{133}$$

$$t > 30 \quad , \quad CPR = 0.06 \tag{134}$$

Notice that monthly mortality rates are lower when the mortgage is less than 30 months old. Mortgages are prepaid less early in their lives. CPR is also a function of the difference between the current rate on the mortgage contract and the going rate for new mortgages (refinancing). Also in summer months, homeowners often move, sell the collateral and payoff their existing mortgages

There are variations of the PSA prepay model. In the 150% (50%)PSA model CPR is simply 1.5 (0.5) times the CPR of the PSA 100% model.

**MBS Pricing and Quoting**

The prices of an MBS are quoted as a percentage of the underlying mortgage balance. The mortgage balance at time-t, $F_t$ is quoted as a proportion of the original balance. This is called the pool factor $pf_t = F_t/F_0$.

Suppose an MBS backed by a collateral pool originally worth \$100M, a current $p_f$ of 0.92 and quoted at 95-16 (-16 is 16/30=50 cents) would have a market value of \$87.86.

$F_t = pf_t \times F_0$ means that $F_t = 0.92 \times 100M = 92M$ so that the market value of the MBS is $0.9950 \times 92 = 87.86$.

MBS values are sensitive to interest rates in two ways: First the discount rates change second the distribution of cashflows change. If rates fall, discount rates are lower and cash flows are earlier since people prepay; both effects increase the value of the MBS. if rates increase, discount rates are higher and cash flows are later; both effects decrease the value of the MBS. Because of this double sensitivity MBS prices are more sensitive to interest rates than corporates.

There are other factors that effect prepayments:

1) Housing turnover: Homeowners often prepay when they move.

2) Credit Curing: It takes time for a prepay rates in mortgages in a mortgage pool to converge to their long term values because improved credit or increased collateral prices allow better rates or larger loans more early on in the life of a loan.

**Pricing an American Bond Option using the Merton Ho-Lee Model**

In a previous section we have covered the European version. It is straightforward to price the American version.

At each node we ask the question is early exercise optimal. Let $P^1_{t+\Delta}$ and $P^2_{t+\Delta}$ be the two option values at the end of the sub-tree, then to we compute for a American Call Bond option;

$$P_t = max(exp(-r_t \times \Delta) \times 0.5 \times \left[P^1_{t+\Delta} + P^2_{t+\Delta}\right], max(B_t - K, 0)) \tag{135}$$

and

$$P_t = max(exp(-r_t \times \Delta) \times 0.5 \times \left[P^1_{t+\Delta} + P^2_{t+\Delta}\right], max(K - B_t, 0)) \tag{136}$$

for a American Put Bond Option. We always make sure that $B_t$ is the price of a three month bond.

holeemertontreeamericanoption.m does the pricing of an american call or a put on a T-bill using a calibrated holee tree.

It is easy to make sure that $B_t$ always gives the value of a 3 month bill for a time-homogeneous model (sice $B_t$ is only a function of $r_t$ and a model parameters plus the maturity parameter which is 3 months).

In a time inhomogeneous model like Cont-Holee or the BDT model, $B_t$ will depend on $t, r_t$, model parameters plus the maturity parameter. This makes the computation slightly more involved.

bdttreeamericanoption.m does the pricing of an american call or a put on a T-bill using a calibrated BDT tree.

**Going From the Risk Neutral Measure to the Physical Measure**

Consider the simplest model we have considered, the Merton Model: Under the risk neutral measure;

$$dr = \alpha dt + \sigma d\hat{B}_t \tag{137}$$

. So the risk neutral (pricing) measure drift is $\alpha$ and the risk neutral standard deviation is $\sigma$.

We had an analytical expression for the bond prices;

$$\Lambda_{t,T} = exp(-(T-t)r_t - \alpha(T-t)^2/2 + 1/2 \times \sigma^2 \times (T-t)^2/3) \tag{138}$$

Notice that the continuously compounded yields are given by

$$y_{t,T} = -log(\Lambda_{t,T})/(T-t) = r_t + \alpha(T-t)/2 - 1/2 \times \sigma^2 \times (T-t)/3 \tag{139}$$

or exploiting time-homogeneity we can write the yields on a $\tau$ maturity bond.

$$y_{t,\tau} = -log(\Lambda_{t,T})/(T-t) = r_t + \alpha\tau/2 - 1/2 \times \sigma^2 \times \tau/3 \tag{140}$$

How do we estimate $\alpha$ and $\sigma$ using Maximum Likelihood given that we historically observe the 3 month bond yield though time?

We can rewrite equation 140 as

$$r_t = y_{t,\tau} - \alpha\tau/2 + 1/2 \times \sigma^2 \times \tau/3 \tag{141}$$

Suppose we have a time series of $y_\tau^t$ for t=1,...,T, i.e. T observations thought time (say the first column of the fama bliss yields).

We also know that in the merton model in the risk neutral measure;

$$r_{t+\Delta}|r_t \sim N(r_t + \alpha\Delta, \sigma^2\Delta) \tag{142}$$

which implies that

$$y_{t+\Delta,\tau}|y_{t,\tau} \sim N(y_{t,\tau} + \alpha\Delta, \sigma^2\Delta) \tag{143}$$

So we can presumably perform a maximum likelihood estimation by writing the likelihood function as

$$L(r_T, r_{T-1}, ..., r_3, r_2|r_1\alpha, \sigma) = f(r_T|r_{T-1}) \times f(r_{T-1}|r_{T-2} \times \dots f(r_2|r_1) \tag{144}$$

However, we can not use 143 in writing out the likelihood function since the observations happen in the physical measure (data generating measure). So we need to go from the risk neutral measure to the physical measure. How do we do this?

It can be shown that there is a connection between the physical and risk neutral measures.

In particular, in a one factor model, let $\mu(r_t, t)$ be the risk neutral drift, and let $\sigma(r_t, t)$ be the risk neutral volatility of the short rate. Then, in the physical measure the drift is $\mu(r_t, t) - \sigma(r_t, t) \times \Lambda(r_t, t)$, where $Lambda(r_t, t)$ is any positive valued function of $r_t$ and $t$ (under some regularity conditions). $Lamda(r_t, t)$ is called the market price of risk. The volatility is the same for the risk neutral and the physical measures. Lets assume constant price of risk at $\lambda$. We can write the physical measure dynamics of the short rate as follows:

$$dr_t = [\alpha - \sigma \times \lambda]dt + \sigma d\hat{B}_t \tag{145}$$

Notice that this specification does not allow separate identification of the risk neutral drift and the market price of risk.

In other words the optimizer will only settle on $\alpha - \sigma \times \lambda$ and $\sigma$ , not $\alpha, \sigma, \lambda$.

Nevertheless we can define $\Gamma = \alpha - \sigma \times \lambda$ and estimate $\sigma$ and $\Gamma$ using Maximum Likelihood estimation (MLE).

mertonmle.m estimates the mertonmodel using maximum likelihood on the fama-bliss data (using annual yields).

Lets perform MLE on the vasicek model and estimate it on the fama bliss data set.

Remember that in the Vasicek model, under the risk neutral dynamics,

$$dr = A(\bar{x} - r_t)dt + \sigma d\hat{B}_t \tag{146}$$

.

Lets assume that the market price of risk is $\Lambda_t = \lambda$, a constant.

Then under the physical measure

$$dr = A(\bar{x} - r_t) - \lambda\sigma dt + \sigma d\hat{B}_t \tag{147}$$

we can rewrite

$$dr = A(\bar{x} - \lambda\sigma/A - r_t)dt + \sigma d\hat{B}_t \tag{148}$$

Relabelling;

$$dr = A(\mu - r_t)dt + \sigma d\hat{B}_t \tag{149}$$

Notice that in the physical measure, equation 149 implies;

$$r_{t+\Delta}|r_t \sim N(r_t e^{-\Theta\Delta} + \mu(1 - e^{-\Theta\Delta}), \sigma^2/(2\Theta)e^{-2\Theta\Delta} \times (e^{2\Theta\Delta} - 1)) \tag{150}$$

First, the $r_t$'s are backed out from the yield. vasicekmle.m estimates the vasicekmodel using maximum likelihood on the fama-bliss data (using annual yields).

Final Examination (due March 3rd, 2008)

1) Consider the Mortgage Backed Security example we covered in class. Modify the mbs.m such that it provides the cash flows of the 150% PSA case. Then price interest and principle (including prepayments)cash flows separately off the current term structure from Bloomberg assuming they are certain to happen.

2) Consider the hedging example where we considered hedging a bond option in the BDT set up. Perform the same exercise but now for the Merton-Ho-Lee tree set-up.

3) Consider the Bill futures analysis we covered in class. Perform the same using a calibrated Continuous Ho-Lee tree rather than the BDT tree.

4) Estimate the simplified merton model by Maximum Likelihood using the 1 year rates in the Fama-Bliss data set. Can you identify the market price of risk?