

An Explicit, Multi-Factor Credit Default Swap Pricing Model with Correlated Factors

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Abstract

With the recent significant growth in the single-name credit default swap market has come the need for accurate and computationally efficient models to value these instruments. While the model developed by Duffie-Pan-Singleton (2000) model can be used, the solution is numerical (solving a series of ordinary differential equations) rather than explicit. In this paper, we provide an explicit solution to the valuation of a credit default swap when the interest rate and the hazard rate are correlated by using the “change of measure” approach and solving a bivariate Riccati equation. CDS transaction data for the period 2/15/2000 through 4/8/2003 for 60 firms are used to both test the goodness of fit of the model and provide estimates of the influence of economic variables in the market for credit risky bonds.

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I. Introduction

Credit default swaps are the newest and fastest growing entrant into the derivatives market. According to the International Swaps and Derivatives Association's year-end 2005 market survey of privately negotiated derivatives, the notional outstanding volume of credit default swaps was \$17.3 trillion, representing an annual growth of 103% in 2005 and 123% in 2004. These swaps are classified as single-name credit default swaps, basket/portfolio default swaps, and credit default swap indexes. Our focus in this paper is on the single-name credit default swap (referred to as CDS hereafter).

The common methodology used to price CDS is the so-called "reduced form" models. Reduced-form models assume that defaults occur unexpectedly and follow a Poisson process. This assumption greatly reduces the complexity since the Poisson process has desirable mathematical properties. In order to further simplify the model, developers of reduced-form models make other assumptions in order to generate closed-form solutions to value a credit-risky bond. While the recovery amount can fluctuate randomly over time because it depends on the liquidation value of the firm at the time of default, simplifications are made in the two leading reduced-form models, the models developed by Jarrow and Turnbull (1995) and Duffie and Singleton (1999). The Jarrow-Turnbull model assumes a recovery in the event of default of a fixed amount and is therefore referred to as a *fixed-recovery model*. Duffie and Singleton (1999) assume the amount of recovery is restricted to be the proportion of the bond price at the time of default as if it did not default and is therefore referred to as a *fractional recovery model*. The rationale is that as the credit quality of a bond deteriorates, the price falls. At default, the recovery price will be some fraction of the final price immediately prior to default. This avoids the contradictory scenario which can arise in the Jarrow-Turnbull model in which the recovery rate, being an exogenously specified percentage of the default-free payoff, may actually exceed the price of the bond at the moment of default.

Duffie, Pan, and Singleton (2000) provide a general approach for valuing any type of default-free and default-risky asset. As a result, their approach allows for the general pricing of CDS. However, their approach does not provide for a closed-form solution but rather a solution by numerical methods.¹ More specifically, the solution requires solving a series of ordinary differential equations. They argue that currently available software will allow for fast computation. However, the problem is that as the number of factors in a model increases, the computational efficiency deteriorates exponentially. In the case of CDS, the factors are the risk-free rate and the default rate (i.e., hazard rate). Each rate, in turn, can depend on a number of factors.

In practice, the CDS pricing models that use the Duffie-Pan-Singleton approach cannot estimate all the parameters in the model simultaneously because of the associated computational difficulties. Instead, they compute some parameters outside the model empirically, such as the correlation between the factors. In this paper, we provide an explicit solution to the value of a CDS by using the “change of measure” approach and solving a bivariate Riccati equation.² As a result, the computational efficiency of our model is only minimally affected by the number of factors. Because our model can be solved quickly, it can simultaneously estimate all the parameters of the model, thereby providing two advantages over the models currently used that are based on the Duffie-Pan-Singleton approach.

To test our model, we use transaction data from Creditex for the period 2/15/2000 through 4/8/2003 to study 60 companies, split evenly between industrial and financial companies. We estimate a two-factor model for the risk-free rate and a one-factor model for the hazard rate. For each of the three factors (two risk-free rates and the hazard rate), there are four parameters: mean-reversion, reverting level, volatility, and market price of risk. There are also two

¹ Other explicit solutions exist under more restrictive assumptions, such as independence between factors (e.g., Longstaff, Mithal, and Neis (2005)) and perfect correlation between the risk-free rate and the hazard rate (e.g., Bakshi, Madan, and Zhang (2002)).

² For an explanation of the use of the bivariate Riccati equation in financial modeling, see Boyle, Tian, and Guan (2002).

correlations with respect to interest rate factors that must be estimated. Consequently, in total, our model requires that 14 parameters be estimated simultaneously. Our results indicate that the three-factor model performs reasonably well in fitting the CDS spreads for almost all of our sample companies. As a byproduct of our testing of the model, we investigate with respect to the credit (i.e., hazard) parameters (1) whether there are differences between industrial and financial companies and (2) their connection to agency ratings.

The paper is organized as follows. The model includes two components: a model of the term structure of the risk-free rate (i.e. interest rate model) and a model of the hazard rate (i.e., the credit model). In addition, there is correlation structure that must be modeled for all the factors in the interest rate and credit models. Section II explains the interest rate model used while Section III describes both the credit model and the solution for the pricing of CDS contracts. The empirical work is provided in Sections IV, V, and VI, with Section IV describing the transaction data and other variables used in the study. The empirical analysis consists of two parts. In the first part, we test for how well our model fits the transaction data. This is provided in Section V. The economic interpretation of the results of the parameters of our estimated model is the focus of the second part of the empirical analysis and is described in Section VI. As will be seen, our model allows us to endogenously estimate the correlation and the default risk premia of our sample data. This has not been done in previous studies because prior studies have taken the parameters as exogenous to the model. Our conclusions are summarized in Section VII.

There is a lengthy appendix (Section VIII) that covers several areas. There are three propositions that are stated in the paper. The mathematical derivation of these propositions is provided in Sections VIIIA, VIIIB, and VIIC. In Section VIID, we provide a short derivation for the forward measure. Section VIIIE provides both a step-by-step illustration of how to use our model and efficiency comparisons versus the Duffie-Pan-Singleton model to value a risk-free bond. The unscented Kalman filter methodology that is used to estimate our model is described in Section VIIF.

II. Interest Rate Model

In developing the interest rate model, as well as the credit model described in Section III, we must assure that interest rates and the hazard cannot be negative. The approach we adopt is a quadratic term structure model (e.g., Constantinides (1992), Ahn, Dittmar and Gallant (2002), and Leippold and Wu (2002)) with a time-dependent drift adjustment (e.g. Scott (1996) and Chen and Yang (2003)). We specify a set of n factors that jointly determine the instantaneous risk-free rate:

$$(1) \quad r(t) = \sum_{i=1}^n [\beta_i(t) + y_i^2(t)]$$

where $r(t)$ represents the instantaneous risk-free rate, $\beta_i(t)$ is time-dependent function, and $y_i(t)$ follows a mean-reverting Gaussian process:

$$(2) \quad dy_i(t) = -\alpha_i y_i(t)dt + \sigma_i dW_i(t), \quad \text{for } i = 1, \dots, n$$

where the α_i 's and σ_i 's are constants, and $W_i(t)$'s are correlated Wiener processes (i.e., $dW_i dW_j = \rho_{ij} dt$).

In the matrix notation, we denote:

$$Y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} \quad \text{and} \quad A_n = \begin{bmatrix} -\alpha_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\alpha_n \end{bmatrix}$$

We further denote

$$\Omega_n = \begin{bmatrix} \omega_{1,1} & 0 & \cdots & 0 \\ \omega_{2,1} & \omega_{2,2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \omega_{n,1} & \omega_{n,2} & \cdots & \omega_{n,n} \end{bmatrix}$$

so that the correlation, ρ_{ij} between state variables i and j , can be mapped into the following variance-covariance matrix³

³ This is known as the Chelosky decomposition.

$$\Omega_n^\top \Omega_n = \Sigma_n = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \rho_{n-1,n}\sigma_{n-1}\sigma_n \\ \rho_{1n}\sigma_1\sigma_n & \rho_{2n}\sigma_2\sigma_n & \cdots & \sigma_n^2 \end{bmatrix}$$

There are two approaches that can be taken with respect to the risk-free term structure. The first is to take the term structure as given and build on that model the credit component (i.e., hazard rate model) without the need for solving for the risk-free term structure. If we take the risk-free term structure as given, then there is no need to specify a specific model. This is the approach employed by, for example, Longstaff, Mithal, and Neis (2005). The drawback of this approach is that one cannot correlate the factors in the interest rate and credit models; that is, it assumes the factors are independent. The second approach is to build the interest rate model in which we correlate all the factors. This is the tact we employ.

Proposition 1 gives the closed-form solution for the risk-free discount factor. The lengthy proof of this proposition is provided in the appendix (Section VIIIA).

[Proposition 1] *The pricing model for the risk-free term structure of interest rates is given as follows:*

$$\begin{aligned} P(0, t) &= E_Y \left[\exp \left(- \int_0^t r(u) du \right) \right] \\ &= E_Y \left[\exp \left(- \int_0^t \sum_{i=1}^n \beta_i(u) + Y^\top(u) Y(u) du \right) \right] \\ (3) \quad &= \tilde{E}_Y \left[\exp \left(- \int_0^t \sum_{i=1}^n \beta_i(u) \right) \frac{d\mathbb{P}}{\tilde{d\mathbb{P}}} \exp \left(- \int_0^t Y^\top(u) Y(u) du \right) \right] \\ &= \exp \left(- \frac{1}{2} Y^\top(0) \Gamma(0, t) Y(0) \right) \exp \left(- \frac{1}{2} \int_0^t \sum_{i=1}^n \beta_i(u) + \text{tr}[\Omega^\top \Gamma(u, t) \Omega] du \right) \end{aligned}$$

where

$$\Gamma_n(t) = \begin{bmatrix} \gamma_{1,1}(t) & \cdots & \gamma_{1,n}(t) \\ \vdots & \ddots & \vdots \\ \gamma_{n,1}(t) & \cdots & \gamma_{n,n}(t) \end{bmatrix}$$

is the solution to the Riccati equation given by [11] in the Appendix (Section VIIIA).

In Section IV where we present the empirical study, we map this solution to the Cox, Ingersoll, and Ross (1985) model (CIR model) to guarantee that the risk-free rate cannot be negative.

III. Credit Model

To understand our credit model, it is necessary to review the two basic economic elements of a CDS contract. A CDS contract follows the swap trading convention of having two legs – a protection leg and a premium leg. The “buyer” of the CDS contract who buys protection against default must pay periodically a fixed spread each period (usually quarterly) as a “premium leg payer.” (This is just like the swap fixed leg payer who pays a fixed swap rate in an interest rate swap.) In return, if default occurs, the “seller” of the CDS contract must compensate the “buyer” for the loss (equal to face value minus recovery value). As a result, the protection leg contains only one probabilistic cash flow when default occurs. (This is similar to the floating leg in a fixed for floating interest rate swap.) The protection value of a CDS contract describes the value of the default protection. This is the dollar value the CDS buyer would need to pay if the buyer would pay for the default protection in a lump-sum amount.

Below we provide a solution to the premium leg and protection leg of a CDS contract.

A. Premium Leg

We begin by augmenting our factor space to include default-risky factors (i.e., the factors that expand the hazard rate). We specify an additional set of ℓ factors that jointly impact the hazard rate:

$$(4) \quad h(t) = \sum_{i=1}^{\ell} [b_i(t) + z_i^2(t)]$$

where $h(t)$ represents the instantaneous hazard rate, $b_i(t)$ is a time-dependent function, and $z_i(t)$ is assumed to follow a mean-reverting Gaussian process as follows:

$$(5) \quad dz_i(t) = -\alpha_{n+i} z_i(t) dt + \sigma_{n+i} dW_{n+i}(t), \quad \text{for } i = 1, \dots, \ell$$

In matrix form that is more convenient to present our model, we let

$$X(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \\ z_1(t) \\ \vdots \\ z_\ell(t) \end{bmatrix} \quad \text{and} \quad A_{n+\ell} = \begin{bmatrix} -\alpha_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\alpha_{n+\ell} \end{bmatrix},$$

$$\Omega_{n+\ell} = \begin{bmatrix} \omega_{1,1} & 0 & \cdots & 0 \\ \omega_{2,1} & \omega_{2,2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \omega_{n+\ell,1} & \omega_{n+\ell,2} & \cdots & \omega_{n+\ell,n+\ell} \end{bmatrix}$$

so that the correlation, ρ_{ij} between state variables i and j , can be mapped into the following variance-covariance matrix⁴

$$\Omega^\top \Omega = \Sigma = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n+\ell}\sigma_1\sigma_{n+\ell} \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \rho_{n+\ell-1,n+\ell}\sigma_{n+\ell-1}\sigma_{n+\ell} \\ \rho_{1n+\ell}\sigma_1\sigma_{n+\ell} & \rho_{2n+\ell}\sigma_2\sigma_{n+\ell} & \cdots & \sigma_{n+\ell}^2 \end{bmatrix}$$

Given the above, Proposition 2 below sets forth the solution to the premium leg. The solution is derived in the appendix (Section VIII B)

[Proposition 2] *For a fixed spread, s , the premium leg is given as follows:*

$$(6) \quad s \sum_{j=1}^N V(0, T_j)$$

where

$$\begin{aligned} V(0, t) &= E_{Y,Z} \left[\exp \left(- \int_0^t (r(u) + h(u)) du \right) \right] \\ &= \exp \left(- \frac{1}{2} X^\top(0) \Gamma(0, t) X(0) \right) \exp \left(- \frac{1}{2} \int_0^t \left(\sum_{i=1}^{\ell} \beta_i(u) + \sum_{i=1}^n b_i(u) + \text{tr}[\Omega^\top \Gamma(u, t) \Omega] \right) du \right) \end{aligned}$$

is known as the risky discount factor and

⁴ This is known as the Chelosky decomposition.

$$\Gamma_{n+t}(t) = \begin{bmatrix} \gamma_{11}(t) & \cdots & \gamma_{1,n+t}(t) \\ \vdots & \ddots & \vdots \\ \gamma_{n+t,1}(t) & \cdots & \gamma_{n+t,n+t}(t) \end{bmatrix}$$

is the solution to the Riccati equation given by [11] in the Appendix (Section VIII B)

Similar to the risk-free discount factor introduced in Proposition 1, $P(0, t)$, that provides the present value of \$1 paid at time t , the risky discount factor in Proposition 2, $V(0, t)$, computes the present value of \$1 paid at time t if there is no default and \$0 if there is default. As explained above, this premium leg is an important component in pricing CDS in that we need to discount the periodic swap payments (i.e., the CDS spreads) made by the CDS buyer.⁵

If we separate the risk-free component from the premium leg, as many market participants do, we arrive at the “forward-measure” survival probability (see Chen and Huang (2001)) as follows:

$$\begin{aligned} V(0, t) &= E_{Y,Z} \left[\exp \left(- \int_0^t (r(u) + h(u)) du \right) \right] \\ &= E_Y \left[\exp \left(- \int_0^t r(u) du \right) \right] E_{Y,Z} \left[\frac{\exp \left(- \int_0^t r(u) du \right)}{E_Y \left[\exp \left(- \int_0^t r(u) du \right) \right]} \exp \left(- \int_0^t h(u) du \right) \right] \\ (7) \quad &= E_Y \left[\exp \left(- \int_0^t r(u) du \right) \right] E_{Y,Z} \left[\frac{d\hat{\mathbb{P}}^t}{d\mathbb{P}} \exp \left(- \int_0^t h(u) du \right) \right] \\ &= E_Y \left[\exp \left(- \int_0^t r(u) du \right) \right] \hat{E}_Z^t \left[\exp \left(- \int_0^t h(u) du \right) \right] \\ &= P(0, t) Q(0, t) \end{aligned}$$

where $P(0, t)$ represents the risk-free discount factor and $Q(0, t)$ is the survival probability under the forward measure.⁶ A short derivation of the forward measure is given in the Appendix (Section VIII D). If the hazard rate and the risk-free rate are independent, then $Q(0, t)$ is the survival probability under the risk-neutral measure.

B. Protection Leg

⁵ It is also used to discount coupons of a default-risky fixed rate coupon bond.

⁶ For the forward measure and the separation of the expectation, see Jamshidian (1987) and Hull (2003).

Now we turn to the valuation of the protection leg. Proposition 3 gives this value, with the proof being provided in the appendix (Section VIII C).

[Proposition 3] *The protection value of a CDS is*

$$\begin{aligned}
D(0, T) &= E_{Y, Z, \tau} \left[1_{\tau < T} \exp \left(- \int_0^\tau r(u) du \right) \right] (1 - \xi) \\
&= (1 - \xi) E_{Y, Z} \left[\int_0^T \left(\sum_{i=1}^{\ell} b_i(\tau) + Z^\top(\tau) Z(\tau) \right) \exp \left(- \int_0^\tau \left(\sum_{i=1}^{\ell} b_i(u) + Z^\top(u) Z(u) \right) du \right) \times \right. \\
(8) \quad &\quad \left. \exp \left(- \int_0^\tau \left(\sum_{i=1}^n \beta_i(u) + Y^\top(u) Y(u) \right) du \right) d\tau \right] \\
&= (1 - \xi) \int_0^T \left[\left(\sum_{i=1}^{\ell} b_i(\tau) + \Psi(\tau) \begin{bmatrix} 0_{n \times 1} \\ 1_{\ell \times 1} \end{bmatrix} \right) \times \right. \\
&\quad \left. \exp \left\{ - \frac{1}{2} X^\top(0) \Gamma(0, \tau) X(0) - \int_0^\tau \left(\sum_{i=1}^{\ell} b_i(u) + \sum_{i=1}^n \beta_i(u) + \frac{1}{2} \text{tr}[\Omega^\top \Gamma(u, \tau) \Omega] \right) du \right\} \right] d\tau
\end{aligned}$$

where ξ is the fixed recovery rate.

Not only does Proposition 3 compute the protection value of a CDS, it also describes the present value of recovery paid upon default. Hence, we can easily derive the fixed-rate coupon bond pricing model in the following Corollary.

[Corollary] *We can use Proposition 2 and 3 to price risky coupon bonds:*

$$(9) \quad B = \sum_{i=1}^N V(0, T_i) c(T_i) + \frac{\xi}{1 - \xi} D(0, T_N)$$

where $c(T_i)$ is the cash flow occurring at time T_i .

C. Solving for the Marked-to-Market CDS Spread

In Section IV we will empirically test our model using CDS spreads when a CDS contract is first traded. That spread is called the market-to-market (MTM) CDS spread and is the spread that will produce a value for the premium leg that is equal to the value of the protection leg so that no cash is exchanged between the counterparties at the inception of the transaction. Hence, we need the solution to the MTM CDS spread for our empirical analysis.

Recall that by the swap convention, the “buyer” of a swap contract will pay a series of fixed payments in exchange for random (in the case of a CDS, also probabilistic) payments. Proposition 3 provides the value of the protection leg. The series of fixed payments constitutes a default-risky annuity which has a present value, by using Proposition 2, $s \sum_{j=1}^N V(0, T_j)$, where s represents the fixed payment and T_j represents the time of the j -th payment. At inception of a swap contract, the fixed spread s is set so that the two legs are equal. Formally, the spread is computed as:

$$(10) \quad s(0, T) = \frac{D(0, T)}{\sum_{j=1}^N V(0, T_j)}$$

IV. Data

The CDS data analyzed in this paper are provided by Creditex and consist of CDS spreads of 1,372 reference entities for the period 2/15/2000 to 04/08/2003. For each reference entity, we obtained the credit rating from Standard & Poor’s, as well as its sector information from Reuters (publicly available on Yahoo). The data contain 217,478 observations (98,971 bids, 85,639 asks, and 32,867 trades) across five currencies (U.S. dollars, Japanese yen, euros, Australian dollars, and British pounds), with the U.S. dollar being the majority of the observations (76.5%), followed by euros (19.1%). The reference entities are classified into four sectors (industrial, financial, sovereign, and telecom) with industrial and financial accounting for 92% of the data.⁷ The majority of the CDS data are for the protection of senior unsecured debts (87.2%). The observations belong to 1,372 names, with 730 (53.2%) being U.S. companies and 457 (33.3%) European companies.

We find that the most liquid names in the dataset are financial firms, with Ford Motor Credit Company being the most traded (1,069 bids, 925 asks, and 355 trades). The most liquid name in the industrial sector is Daimler Chrysler AG. (1,146 bids, 1,086 asks, and 262 trades). The liquidity drops substantially if we move out of the top 50 names. For example, the 15-th

⁷ The classification of the sectors in the data is coarse. For example, finance.yahoo.com provides further breakdowns within the industrial sector and the financial sector.

most liquid name in the industrial sector is McDonald's Corporate and it has less than half of the bids and asks of Daimler Chrysler AG (320 bids and 268 asks), and only 48 trades. For this reason, several filters are imposed to construct the sample CDS. First, though there are four different sectors in the dataset, we focus on the two main sectors: industrial (931 entities) and financial (332 entities). Second, since our estimations only use trading data (not bids or asks), based on trading frequency, the most liquid 30 industrial reference entities and 30 financial reference entities are included in the sample, leaving us with a dataset of 60 reference entities. Third, only 5-year CDSes have been included in our sample data. Finally, for those firms that have multiple trades in the same trading day, an average spread is taken and used for that trading day. Summary statistics of the MTM CDS spreads for these 60 reference entities are reported in Table 1, with financial firms and industrial firms shown in Panels A and B, respectively.

The spreads differ substantially across firms, even within the same rating/sector class. For example, the lowest mean spread in single A Financial is Bear Stearns (68.42) and the highest is Household Finance Corp. (311.82). Furthermore, there is significant overlap in spreads across groups. For example, the lowest single A Financial (Bear Stearns, 68.42) has a narrower spread than the highest A+ Financial (JP Morgan Chase, 80.57). The two principal reasons for this, as cited widely in the literature, are (1) the slow reaction of the rating agencies in releasing rating changes and (2) credit spreads are not entirely attributable to credit risk (e.g., they are impacted by liquidity and taxes). Nevertheless, due to lack of any better measure for ratings and any benchmark for credit risk,⁸ we continue to use the current ratings as an indicative measure of the credit quality of the firm.

Given that CDS spreads are often benchmarked to LIBOR and the swap rate,⁹ we also obtained from Bloomberg the U.S. dollar LIBOR and swap rates that match the sample period of

⁸ While many studies (e.g. Eom, Helwedge, and Huang (2004)) use the structural models to gauge the spread that contains only credit risk, no consensus has been reached.

⁹ Theoretically, LIBOR is default risky since it is an inter-bank rate. However, many dealers that trade CDS contracts have an equivalent or worse credit quality than LIBOR. Longstaff, Mihal, and Neis (2005) demonstrate that various benchmark rates make little difference in explaining the dynamics of the CDS spreads.

the CDS spread data. We use 12-month LIBOR as well as the 2-year, 3-year, 4-year, 5-year, 7-year, 10-year, 15-year, and 30-year swap rates.

In the next two sections, we present our empirical results. The first part of our empirical analysis tests the goodness of fit of our model. We simultaneously estimate all the parameters of our model using the 60 most liquid CDS contracts, something that could not be accomplished in prior studies because the value of these economic variables were exogenous to the model. In the second part, we try to understand the economics of the estimated parameters implied by the CDS prices. We identify certain patterns of the parameters across financial and industrial sectors.

V. Estimation Results and Goodness of Fit

To estimate our model, we cast it into a state-space model and use the unscented Kalman filter to estimate the parameters and factor values. The maximum likelihood estimation is used to estimate parameter values. This procedure allows us to estimate parameters used in the stochastic processes.

A. *Estimation of the Interest Rate Model*

To price a CDS, we must obtain the benchmark risk-free interest rate information. We derived our model under the risk-neutral measure and therefore the analytical solution for the CDS price allows for a full econometric estimation of the parameters, including the risk premia.

Proposition 1 (equation (3)) demonstrates that the risk-free term structure can be solved analytically when the factors are correlated. In our empirical study, we assume the factors within the risk-free rate to be independent and focus on the correlation between the interest rate and default rate. While our model can incorporate any number of correlations with no computational difficulty, by assuming the interest rate factors to be independent of each other, we can link better our term structure estimation to the literature and obtain cleaner insight into the relationship between the interest rate and the hazard rate. If the factors are independent, then the model can be simplified to the following:

$$\begin{aligned}
(11) \quad P(0, t) &= E \left[\exp \left(\sum_{j=1}^n - \int_0^t [\beta_j(u) + y_j^2(u)] du \right) \right] \\
&= e^{-\sum_{i=1}^n \int_0^t \beta_i(u) du} \prod_{i=1}^n E \left[\exp \left(- \int_0^t y_i^2(u) du \right) \right] \quad i = 1, 2 \\
&= e^{-\sum_{i=1}^n \int_0^t \beta(u) du} \left(\prod_{i=1}^n A_i(0, t) \right) e^{-\sum_{i=1}^n y_i^2(0) B_i(0, t)}
\end{aligned}$$

where

$$\begin{aligned}
A_i(0, t) &= \left[\frac{2\zeta_i e^{[(\alpha_i + \zeta_i)t/2]}}{(\alpha_i + \zeta_i)(e^{\zeta_i t} - 1) + 2\zeta_i} \right]^{\frac{1}{2}} \\
B_i(0, t) &= \frac{2(e^{\zeta_i t} - 1)}{(\alpha_i + \zeta_i)(e^{\zeta_i t} - 1) + 2\zeta_i} \\
\zeta_i &= \sqrt{\alpha_i^2 + 2\sigma_i^2}
\end{aligned}$$

To guarantee non-negativity, we must map (11) to the CIR model. In order to do so, we must specify the following relationship:

$$(12) \quad \exp \left(- \int_0^t \beta_i(u) du \right) = \left[\frac{2\zeta_i e^{[(\alpha_i + \zeta_i)t/2]}}{(\alpha_i + \zeta_i)(e^{\zeta_i t} - 1) + 2\zeta_i} \right]^{\frac{2\alpha_i \mu_i}{\sigma_i^2} - \frac{1}{2}}$$

where $\alpha_i = \hat{\alpha}_i + \lambda_i$, $\hat{\alpha}_i$ is the mean-reverting speed under the real measure, λ_i is the market price of risk for the i -th factor $y_i(t)$,¹⁰ and μ_i is the reverting level of $y_i(t)$ under the real measure, such that price of a zero-coupon bond is given by CIR as:

$$(13) \quad P(0, t) = A_1^*(0, t) A_2^*(0, t) \exp(-y_1^2(0) B_1(0, t) - y_2^2(0) B_2(0, t))$$

where

$$A_i^*(0, \tau) = \left[\frac{2\zeta_i e^{(\hat{\alpha}_i + \lambda_i + \zeta_i)\tau/2}}{(\hat{\alpha}_i + \lambda_i + \zeta_i)(e^{\zeta_i \tau} - 1) + 2\zeta_i} \right]^{\frac{2\hat{\alpha}_i \mu_i}{\sigma_i^2}}, \quad i = 1, 2$$

¹⁰ Cox, Ingersoll, and Ross (1985) show that the risk-neutral speed of mean reversion is the real measure speed plus the market price of risk.

Effectively we tweak the time-dependent factor to each of the independent factors so that each factor follows the square-root process. With the link to the parameters in the real measure, we are now able to estimate the model with time-series data.

The estimation is based upon a state-space model with a Kalman filter. In state-space form, we regard the two spot rates as the unobservable state and specify the state propagation using an Euler approximation of statistical dynamics of the interest rate factor. The diffusion equation for the state variable that governs the risk-free term structure dynamics is:

$$(14) \quad dy_i = -\frac{1}{2}\alpha_i y_i dt + \frac{1}{2}\sigma_i dW_i$$

This equation leads to the state-variable propagation equation given by:

$$(15) \quad y_i(t + \Delta t) = \Theta_i y_i(t) + \sqrt{H_i} \varepsilon_i(t)$$

where $\Theta_i = \exp(-\frac{1}{2}\alpha_i \Delta t)$ and $H_i = \frac{1}{4}\sigma_i^2 \Delta t$. It is this pricing formula that is used to compute the measurement equation:

$$(16) \quad \begin{aligned} \text{LIBOR}(y, \tau) &= \frac{100}{\tau} \left(\frac{1}{P(0, \tau)} - 1 \right) + e_{L,1} \\ \text{SWAP}(y, \tau) &= 200 \frac{1 - P(0, \tau)}{\sum_{i=1}^{2\tau} P(0, \frac{i}{2})} + e_{L,2} \\ \text{cov}(e_L) &= R_\sigma \quad e_L = \begin{Bmatrix} e_{L,1} \\ e_{L,2} \end{Bmatrix} \end{aligned}$$

A convenient approach to deal with measurement errors is to cast LIBOR and swap rates in a state space augmented by measurement equations that relate the observed prices to the underlying state variables. When the state variables are Gaussian and the measurement equations are linear, the Kalman filter yields the efficient state updates in a least-squares sense. However, in our application, the state propagation equation in (15) is Gaussian linear, but the measurement equation in (16) is nonlinear in the state variables. Traditional literature uses an extended version of the Kalman filter (EKF) by approximating the nonlinearity via a Taylor expansion. However,

since the ultimate objective is to obtain the posterior distribution of the state variables given the observations, Julier and Uhlmann (1997) propose the Unscented Kalman Filter (UKF) to directly approximate the posterior density using a set of deterministically chosen sample points. These sample points capture the true mean and covariance of the Gaussian state variables. This approach is computationally efficient because it avoids the calculation of derivatives for the linear approximation. The method also improves the accuracy of the estimates because it reduces the convexity bias induced in the first-order approximation in the EKF. We discuss the UKF and its application to our estimation process in the Appendix (Section VIII F).

From the UKF we obtain efficient forecasts on the conditional mean \bar{y}_t and conditional covariance matrix \bar{V} of the LIBOR/swap rates, and build the likelihood function based on the conditional density of the pricing errors:

$$(17) \quad \max_{\Theta} L_r(\Theta; \{l_t\}) = \sum_{t=1}^N \left[-\frac{1}{2} \log |\bar{V}_t| - \frac{1}{2} \left((l_t - \bar{l}_t)' \bar{V}_t^{-1} (l_t - \bar{l}_t) \right) \right]$$

$$\Theta = \{\hat{\alpha}_1, \mu_1, \sigma_1, \lambda_1, \hat{\alpha}_2, \mu_2, \sigma_2, \lambda_2\}$$

where l_t is a vector of nine LIBOR/swap rates in the dataset at time t .

The estimation results of the interest rate process using LIBOR and swap rates are presented in Table 2.¹¹ The results are similar to what has been reported in the literature. Table 3 reports the summary statistics for the pricing errors of LIBOR and swap rates under the two-factor CIR model. On average, this model only generates 3.4% of root mean square error (RMSE). Model-explained percentage variance (denoted by VR in the table) indicates that the two-factor CIR model performs better for short-term maturities compared to the long-term maturities.

B. Estimation the CDS Spread

¹¹ We also estimated the one factor model for the risk free term structure. The root mean squared errors are significantly worse than those of the two factor model. The results are available from the author on request.

We now take the estimated interest rate model from the previous Section VA as given and estimate the dynamics of the hazard rate (i.e., credit model) using the observed CDS spreads. Under our model specification, the instantaneous hazard rate follows a CIR process too. Under a single-factor model for the credit risk ($\ell = 1$), equations (7) and (8) are greatly simplified as follows:

$$(18) \quad D(0, T) = (1 - \xi) \int_0^T \left[\left(b(\tau) + z^2(\tau) \right) \exp \left\{ -\frac{1}{2} X^\top(0) \Gamma(0, \tau) X(0) - \int_0^\tau \left(b(u) + \beta_1(u) + \beta_2(u) + \frac{1}{2} \text{tr}[\Omega^\top \Gamma(u, \tau) \Omega] \right) du \right\} \right] d\tau$$

and

$$(19) \quad V(0, t) = \exp \left(-\frac{1}{2} X^\top(0) \Gamma(0, t) X(0) \right) \exp \left\{ -\int_0^t \left(b(u) + \beta_1(u) + \beta_2(u) + \frac{1}{2} \text{tr}[\Omega^\top \Gamma(u, t) \Omega] \right) du \right\} \\ = \prod_{i=1}^3 \left[\frac{2\zeta_i e^{[(\alpha_i + \zeta_i)t/2]}}{(\alpha_i + \zeta_i)(e^{\zeta_i t} - 1) + 2\zeta_i} \right]^{\frac{2\alpha_i \mu_i}{\sigma_i^2} - \frac{1}{2}} \exp \left(-\frac{1}{2} X^\top(0) \Gamma(0, t) X(0) - \frac{1}{2} \int_0^t \text{tr}[\Sigma^\top \Gamma(u, t) \Sigma] du \right)$$

This above closed-form solution enables us to estimate the parameter more easily with the UKF. The hazard rate process is estimated in a similar manner. The state-space model is given as:

$$(20) \quad z(t + \Delta t) = \Xi z(t) + \sqrt{U} \varepsilon(t)$$

which is similar to equation (15) where $\Xi = \exp(-\frac{1}{2} \alpha_3 \Delta t)$ and $U = \frac{1}{4} \sigma_3^2 \Delta t$.

$$(21) \quad s(0, T) = \frac{D(0, T)}{\sum_{j=1}^N V(0, T_j)} + e_C$$

for an arbitrary firm.

The pricing model is given by equation (21). We first take the results of the risk-free term structure estimation as given. And recovery rate ξ is 0.4, which is the industry average value. We assume that all of CDS are priced with a normally distributed measurement error and derive the conditional likelihood function of a CDS given the parameters for the interest rate model as:

$$(22) \quad \max_{\{\Psi\}} L_n(\Psi; \{s_t\} | \Theta) = \sum_{t=1}^N \left[-\frac{1}{2} \log |\bar{V}_t| - \frac{1}{2} \left((s_t - \bar{s}_t)^T \bar{V}_t^{-1} (s_t - \bar{s}_t) \right) \right]$$

where $\Psi = \{\hat{\alpha}_3, \mu_3, \sigma_3, \lambda_3, \rho_1, \rho_2\}$, $\Theta = \{\hat{\alpha}_1, \mu_1, \sigma_1, \lambda_1, \hat{\alpha}_2, \mu_2, \sigma_2, \lambda_2\}$.

The estimation of the hazard rate process (i.e., credit model) is performed for each of the 60 firms in the sample. The results, reported in Table 4, vary widely across different firms, even within the same sector/rating group. The mean-reversion parameter (α_3) for the financial firms ranges from 0.349 (Capital One) to 2.496 (International Lease) and for the industrial firms from 0.051 (WorldCom) to 3.059 (Federated Department Stores). The reverting level (μ_3) for the financial firms ranges from 9 basis points (ABN Amro) to 375 basis points (MBNA) and for the industrial firms from 3 basis points (WorldCom) to 1,435 basis points (Vivendi Universal). The volatility (σ_3) for the financial firms ranges from 2.1% (Deutsche Bank) to 314.5% (International Lease) and for industrial firms from 14.7% (McDonald's) to 396.9% (WorldCom). The market price of default risk (λ_3) for the financial firms ranges from -0.317 (Household Finance) to -0.004 (Commerzbank) and for industrial firms from -0.044 (Portugal Telecom) and -0.647 (Computer Associates). Finally, the correlation for the financials between the hazard rate and the first interest rate factor (ρ_1) ranges from $+28.7\%$ (Morgan Stanley) to -88.9% (Ford Motor Credit Company) and the second factor (ρ_2) from $+20.2\%$ (Bank of America) and -50.5% (Deutsche Bank). For the industrial firms, the correlation ranges are the highest $+45.2\%$ (Altria Group) and the lowest -85% (Sprint) with respect to the first factor and the highest $+76\%$ (Daimler Chrysler) and the lowest -47% (Vivendi Universal) with respect to the second factor

Nearly all parameters are significant, clearly indicating that the data demand at least a one-factor model. With only one factor, it is expected that due to company-specific characteristics, these parameters should vary substantially. All parameters carry the right sign and the magnitudes of the value of the parameters are within reasonable ranges.

To examine if the one-factor model has performed satisfactorily, we substitute the parameters into the model to compute both theoretical prices and the absolute pricing error. The results are reported in Table 5. The model generally underestimates the market for the financial sector while overestimates the market for the industrial sector (average error -0.683 for financials

and 0.550 for industrials). The average percentage of the RMSE over the spread is 5.78% and 4.04% for the financial sector and the industrial sector, respectively. The variance ratio (the percentage of the variance explained by the model) is 92.9% for the financial sector and 96.61% for the industrial sector.

The pricing errors can be substantially reduced if we introduce more factors because liquidity and other common factors are not considered in the one-factor hazard rate model.¹² However, if we examine the fitting results of our sample in more detail, we find that mispricing mainly comes from a few outlier observations. To demonstrate this, we plotted the model prices against the transaction prices over time for each firm. To conserve space, we choose eight companies, the four best fits and the four worst fits, from each sector to present in Figure 2. First, we observe a strikingly good fit of the model, even for the companies with large errors. Consider, for example, ABN Amro. Its RMSE is 14.08%. Yet we observe that the mispricing comes from a few observations at the beginning of the trading period in our study. Note that there were no observations prior to September 2001 until the beginning of the sample period; hence, the Kalman filter does not have enough information to provide a good fit. Once observations start to come in, the fit improved and we see very good results. Societe Generale which has a 17.62% RMSE demonstrates this phenomenon. Lack of observations in the beginning of the observation period prevents the Kalman filter from *learning* to provide a good fit. For the industrial sector, the fitting is much better. Despite the better fit, industrial firms such as McDonald's Corp and Walt Disney are the firms with the two worst fits (9.89% and 8.74% errors, respectively), with their fitting errors attributable to one or two points in the history of the sample. From Figure 2, we can conclude that the one-factor model for the credit spread seems to do a suitable job.

VI. Economics of the Estimated Parameters

A. *Parameter Behaviors against Ratings*

¹² One clear piece of evidence of the liquidity problem in CDS quotes is that during this sample period of nearly 800 business days, most of our sample firms have less than 100 trades – about one every two weeks.

While parameter estimates vary substantially, certain expected behaviors are observed. In particular, we observe some interesting similarities and differences between the financial and industrial sectors by plotting the estimates against ratings. Figure 1 plots the average parameter estimates in each rating group against ratings. The left panel plots the average estimated value of the parameter for the financial sector; the right panel does the same for the industrial sector.

The mean-reversion and market price of risk parameters, α_3 and λ_3 respectively, presents identical patterns for the financial and industrial sectors. The patterns for mean-reversion are upward sloping (i.e., lower the credit rating, the faster the mean reversion) while the patterns for the market price of risk are downward sloping (i.e., lower the credit rating, the smaller the market price of risk). The reverting level, μ_3 , and the volatility, σ_3 , of the two sectors both exhibit upward sloping patterns (i.e., the lower the credit rating, the higher the level). However, the financial sector's reverting level is less sensitive to ratings than the industrial sector, while the financial sector's volatility is more sensitive to ratings than the industrial sector. This finding suggests that relative to industrial firms, financial firms are more homogenous in their long-term expected returns (reflected by μ_3), but more heterogeneous in volatility (σ_3). This finding is interesting (and somewhat expected) in that financial firms are highly regulated and homogeneous in nature. The difference resides in their sizes, which is reflected in their volatilities.

The correlations between the hazard rate process and the two factors in the interest rate process, ρ_1 and ρ_2 , exhibit an interesting behavior. Both the financial and the industrial sectors present identical downward sloping patterns in the first correlation, ρ_1 (i.e., the lower the credit rating, the lower the correlation), but very different patterns (i.e., downward sloping for the financial sector but upward sloping for the industrial sector) in the second correlation, ρ_2 . This confirms the fact that industrial firms are more diverse than financial firms, and hence the impact of the correlation is less.

B. Default Risk Premium

A question that has gained increased attention is the magnitude of the default risk premium (DRP) in credit derivative products. In our model, the DRP is measured by the risk premium in the hazard rate dynamics. In our model, this quantity is $\lambda_3 h(t)$, which is consistent with the risk premium of the CIR model. For a risk-neutral individual, $\lambda_3 = 0$ and the cash flows would be discounted at the risk-free rate. For this risk-neutral individual, the instantaneous default probability time is $h^{(\lambda_3=0)} dt$, whereas a risk adverse individual should have a different default probability given by $h^{(\lambda_3 \neq 0)} dt$, holding other parameters unchanged. Some researchers (e.g., see Berndt et al. (2005)) measure the DRP by computing the ratio of the risk-neutral default probabilities and real default probabilities (such as EDFs¹³). We are interested in the portion of the CDS price that is attributable to the DRP.

In Section VB, we estimate the market price of default risk. The DRP embedded in the CDS price (MTM CDS spread) can be calculated as follows:

$$(23) \quad \text{DRP}_i = s_i(0, T | \Theta, \lambda_3 = 0) - s_i(0, T | \Theta, \lambda_3 = \lambda_3^*)$$

where $\Theta = \{\hat{\alpha}_1, \mu_1, \sigma_1, \lambda_1, \hat{\alpha}_2, \mu_2, \sigma_2, \lambda_2; \hat{\alpha}_3, \mu_3, \sigma_3, \lambda_3, \rho_1, \rho_2\}$ and DRP_i is the default risk premium for firm i , $s_i(0, T | \lambda_3 = 0)$ and $s_i(0, T | \lambda_3 = \lambda_3^*)$ are model implied fair value of the CDS given by equation (10).

The results for our sample are reported in Table 6. The first column is the average (over time) of the company's DRP in basis points, the second is the standard deviation, and the last is the percentage of the average MTM CDS spread. As we can see from the table, the average default risk premiums vary substantially for different firms. Consistent with our expectation, higher rated firms present a lower DRP while a higher default DRP is needed for lower rated firms; yet, the relationship is not strong. Financial firms have an average 5.23 basis points DRP that is 4.22% over their spread levels while industrial firms have an average 12.58 basis points DRP or 6.79% over their spread levels. This is consistent with the rating category for these two

¹³ EDF is the Expected Default Frequency, a default probability under real measure used by Moody's KMV.

sectors; that is, financial firms have higher ratings compared to industrial firms in our sample data. The bottom panel of Figure 1 demonstrates graphically the observation in Table 6.

Our observation of DRP is consistent with the finding of Berndt et. al. (2005). They found that there is large time variation of the DRP, although we measure the DRP differently. The standard deviations in Table 6 are quite large compared to the means. The average standard deviation (over time) of the financial sector is 2.99 basis points relative to the average mean of 5.23 basis points. The average standard deviation of the industrial sector is 6.95 basis points relative to the average mean of 12.58 basis points.

Finally, in Figure 1, we plot the DRP against ratings for the financial and industrial sectors. The default risk premiums, like the correlation, are different for the two sectors, exhibiting an upward sloping pattern for the financial sector and a downward sloping pattern for the industrial sector. In general, the financial sector has less default risk premium (an average of about 4% versus 6% for the industrial sector). Moreover, the default risk premiums are positively related to ratings (i.e., the lower the rating, the higher the default risk premium) for the financial sector while it is nearly flat (slightly negatively sloped) for the industrial sector. Again, we interpret this result as an indication that the industrial sector is more diverse than the financial sector.

Recall that the market price of risk is smaller with worsening credit rating. If we roughly interpret the default risk premium as the product of the market price of risk and the quantity of default risk, the result shown in Figure 1 that the DRP is positively related to worsening credit ratings indicates that the quantity of default risk is a highly sensitive function of credit rating. In other words, the quantitative measure of credit risk is a highly non-linear function of indicative credit ratings by the rating agencies.¹⁴

C. Correlation

¹⁴ We thank the referee for making this observation for us.

One unique contribution of this paper, which results from our model, is that we can econometrically and endogenously estimate the correlation between the instantaneous interest rate and the hazard rate. This can be accomplished due to the factor structure and computational efficiency of our model. The parameters therefore can be easily estimated. From Table 4 we find the estimated correlations between the factors of the interest rate and the hazard rate to be significant for most of the firms in the sample. However, the signs are not uniform. While the majority of the firms illustrate negative correlations between the hazard rate and the interest rate, some firms do show positive correlations. This finding indicates that the correlation between the credit risk and the interest rate risk varies across firms, even within the same sector. Furthermore, as Figure 1 demonstrates, both financial firms and industrial firms exhibit no clear correlation pattern in rating (especially ρ_2 where financials exhibit a negative pattern in ratings and industrials exhibit positive).

Table 7 compares the estimated correlation between the hazard rate and the interest rate and the empirical correlation between the MTM CDS spreads and the swap rates. We provide this comparison in order to evaluate the validity of our model. The fourth column in the table represents the empirical correlation between the CDS spreads and the 5-year swap rates ($\rho_{\text{CDS,swap}}$). In general, we find that the correlations are mostly negative (with only seven exceptions: Bear Sterns (12.8%), MBNA (23.8%), Reuters (34.4%), Portugal Telecom (14.8%), Repsol (17%), Computer Associates (14.9%), and Eastman Kodak (24.1%)), indicating that high spreads and low rates usually occur together. The third column is the estimated correlation between the interest rate and the hazard rate ($\rho_{r,h}$). We expect these correlations to be different in magnitude from the empirical ones in that there is a non-linear transformation from the correlation between the interest rate and the hazard rate to the correlation between the CDS spreads and swap rates. However, we should expect the signs to be the same. For the most part, the estimated correlations do carry the same signs as the empirical correlations with the following exceptions: American Express (26.3% theoretical versus -1.2% empirical), Bank One (21% versus -4.9%), Cit Group (8.3% versus -17.7%), Munich Re. (31.6% versus -7.8%), and Boeing (11.3% versus -44.3%). These inconsistencies could imply instability of other parameters since the estimated correlation

is a function of all parameters, combined with too few observations in the sample. Overall, we find that the estimated correlation results are closer to empirical correlations for the financial firms than for the industrial firms, which is shown in Figure 3. The fitted slope between the two correlations reaches a high of 0.95 for the financial firms but only 0.84 for the industrial firms. We also see a higher R-square for the financial firms (65%) than for the industrial firms (61%).

We also examine the relationships between (i) the default risk premiums and the swap rates, reported in column 5 of Table 7, and (ii) the default risk premiums and the CDS spreads, reported in column 6 of Table 7. The default risk premiums are computed by our model as a time series of every company. The correlations between the DRP and swap rates are generally negative (except for Morgan Stanley (80.6%), Munich (80.4%), MBNA (19.5%), Portugal Telecom (34.4%), Repsol (13.8%), and Eastman Kodak (13.5%)) whereas the correlations between the DRP and CDS spreads are overwhelmingly highly positive (except for Morgan Stanley (-81.8%), Munich (-32.3%), and Siemens (-78.7%)). While the negative relationship between the DRP and swap rates is no surprise, the positive relationship between DRP and CDS spreads is interesting. It suggests that as credit quality deteriorates, investors demand more compensation for the same default risk than if it is taken in a better credit quality timeframe.

VII. Conclusion

A general approach for valuing any type of default-free and default-risky asset such as a CDS contract has been proposed by Duffie, Pan, and Singleton (2000). However, the solution to their model is obtained using numerical methods by solving a series of ordinary differential equations. While there is currently available software that allows fast computation, the limitation is that as the number of factors in their model increases, the computational efficiency deteriorates exponentially. Hence, the Duffie-Pan-Singleton approach cannot estimate all the parameters in the model simultaneously because of the associated computational difficulties and, as a result, some parameters must be estimated outside the model. The model we present in this paper for valuing a CDS has an explicit solution. The solution to the model involves solving a bivariate Riccati equation. The two advantages of our model are that it can be solved quickly (the

computation is only minimally affected by the number of factors) and can simultaneously estimate all the parameters of the model.

Using transaction data for MTM CDS spreads, we test the goodness of fit of our model by estimating a three-factor model (two factors for the interest rate and one factor for the hazard rate). The fitting exercises demonstrate that the model does a reasonable job in explaining the time series variation of the spreads. However, an average of 3% pricing errors suggests that there is room for improvement. The parameters estimated reveal interesting economic information with respect to a company's mean reversion, reverting level, volatility, default risk premium, and correlation between the risk-free rate and the hazard rate. By comparing these parameters with ratings, we find that firms in both the financial and industrial sectors have a very similar mean-reversion pattern and reverting-level pattern, a somewhat similar pattern in their risk premium, and very different patterns in their volatility and correlation. This confirms the industry wisdom that credit spreads react to first-order factors more consistently than second-order factors.

Our model can be easily extended to include random recovery. Following Bakshi, Madan, and Zhang (2002), we can assume the recovery as an exponential function of the hazard rate which can be solved similarly by using similar method.

TABLE 1**Summary Statistics of Credit Default Swap Spreads**

This table lists top 30 liquidity industrial firms and top 30 liquidity financial firms based on the transaction updating frequency. Rating information is from Standard & Poor's; Active No. represents the transaction updating frequency during our sample period; Mean, Std, Min, and Max denote the sample mean, standard deviation, minimum and maximum of CDS spreads (in basis points)

Panel A: Top 30 Financial Firms

Name	Rating	# of trades	Mean	Std	Min	Max
AMERICAN EXPRESS	AAA	77	65.71	15.40	35	103
AMERICAN INTERNATIONAL GROUP INC	AAA	43	50.10	15.55	30	92
GENERAL ELECTRIC CAPITAL CORP	AAA	144	65.50	22.47	25	126
ABN AMRO BANK N.V.	AA-	36	30.75	11.11	16	56
BANCO BILBAO VIZCAYA ARGENTARIA S.A.	AA-	40	34.80	14.24	18	67
BANK ONE CORPORATION	AA-	47	39.95	9.86	27	60
CITIGROUP INC.	AA-	120	50.37	19.03	23	110
DEUTSCHE BANK A.G.	AA-	42	41.26	20.50	16	80
FEDERAL NATIONAL MORTGAGE ASSO	AA-	54	26.50	3.89	18	36
SOCIETE GENERALE	AA-	32	33.44	13.69	15	57
BANCO SANTANDER CENTRAL HISPANO SA	A+	74	56.39	24.70	18	145
BANK OF AMERICA	A+	72	41.47	10.86	24	67
GOLDMAN SACHS	A+	128	60.49	13.06	38	95
JP MORGAN CHASE	A+	141	80.57	24.02	27	170
MERRILL LYNCH	A+	131	75.60	31.34	32	226
MORGAN STANLEY	A+	106	60.71	18.39	34	134
BEAR STEARNS CO	A	62	68.42	12.72	46	100
CIT GROUP INC	A	102	238.52	84.09	22	445
COUNTRYWIDE HOME LNS INC	A	61	106.45	24.87	71	185
HOUSEHOLD FINANCE CORPORATION	A	160	311.82	254.18	45	1000
LEHMAN BROTHERS	A	73	70.58	16.84	35	115
BAYERISCHE HYPO-UND VEREINSBANK AG	A-	32	105.13	53.22	24	265
COMMERZBANK A.G.	A-	41	99.06	76.29	13	300
FLEETBOSTON FINANCIAL CORPORATION	A-	59	69.29	29.98	39	130
MUNICH RE	A-	36	59.39	17.17	37	134
MBNA AMERICA BANK, NA	BBB+	63	141.11	26.20	115	220
CAPITAL ONE BANK	BBB	97	469.11	182.88	98	925
MBNA CORPORATION	BBB	113	185.31	50.42	96	300
FORD MOTOR CREDIT COMPANY	BBB-	355	229.47	132.37	55	700
INTERNATIONAL LEASE FIN CORP	BB	57	136.18	15.13	85	175

Panel B: Top 30 Industrial Firms

Name	Rating	# of trades	Mean	Std	Min	Max
E. ON AG	AA-	35	66.71	19.24	24	88
SIEMENS AG	AA-	61	60.93	14.72	36	95
RWE AG	A+	41	74.55	28.68	21	116
VERIZON GLOBAL FUNDING	A+	44	151.01	70.38	82	340
BAYER AG	A	37	87.95	54.79	22	200
BOEING COMPANY	A	62	80.78	25.57	20	130
MCDONALD'S CORP	A	48	50.98	19.55	16	120
REUTERS GROUP PLC	A	37	162.32	38.36	100	230
PORTUGAL TELECOM INT'AL FINANCE BV	A-	55	113.11	31.89	55	180
VOLKSWAGEN AG	A-	44	58.30	19.46	26	110
WORLDCOM INC	A-	31	254.82	158.36	110	675
ALTRIA GROUP INC	BBB+	89	144.13	105.16	29	800
AOL TIME WARNER INC	BBB+	110	244.14	154.84	41	650
KONINKLIJKE (ROYAL) PHILIPS ELEC NV	BBB+	72	107.15	25.77	51	180
REPSOL S.A.	BBB+	44	226.30	116.11	45	550
WALT DISNEY COMPANY	BBB+	106	91.93	32.51	26	190
DAIMLERCHRYSLER AG	BBB	262	137.02	36.08	44	247
HEWLETT PACKARD CO	BBB	50	103.18	29.81	63	175
MOTOROLA INC	BBB	45	290.64	72.76	192	450
SEARS ROEBUCK ACCEPTANCE CORP	BBB	94	226.40	132.92	60	450
COMPUTER ASSOC INTL INC	BBB-	42	329.32	96.26	180	560
DELPHI	BBB-	52	150.13	44.21	87	315
EASTMAN KODAK COMPANY	BBB-	64	137.44	63.77	30	425
FEDERATED DEPARTMENT STORES INC	BBB-	41	103.47	22.88	70	145
SPRINT CORPORAT	BBB-	57	325.75	182.75	109	925
VIVENDI UNIVERSAL S.A	BBB-	62	79.48	26.50	17	130
VISTEON CORPORATION	BB+	35	210.83	63.85	95	331
TOYS "R" US INC	BB	35	286.83	127.24	88	575
FIAT SPA	BB-	118	287.33	175.89	48	850
GENERAL MOTOR	BB-	202	173.73	86.83	64	405

TABLE 2**Risk-Free Term Structure Estimation**

This table reports the first-step parameter estimates of the two-factor risk-free term structure that determine the dynamics and term structure of the benchmark LIBOR and swap rates. The t-statistics are presented in the second row. The estimation is based on 12-month LIBOR and swap rates of 2, 3, 4, 5, 7, 10, 15, and 30 years, using quasi-maximum likelihood method jointed with Unscented Kalman filter.

	$\hat{\alpha}_1$	μ_1	σ_1	λ_1	$\hat{\alpha}_2$	μ_2	σ_2	λ_2
Coef.	0.2582	0.0559	0.2481	-0.1707	2.5938	0.0005	0.9833	-0.0617
t	3.8494	6.6532	4.5140	-5.2999	42.6763	0.5387	12.2910	-7.9700

TABLE 3**Summary Statistics of Pricing Errors on the LIBOR and Swap Rates**

Entries report the summary statistics of the pricing errors on the LIBOR and swap rates under the two-factor model. We estimate this two-factor model with quasi-maximum likelihood method jointed with Unscented Kalman filter. We define the pricing error as the difference between the observed interest rate quotes and model-implied fair values. The columns titled with Mean, Std, ME%, RMSE, RMSE%, VR denote the mean error, the standard deviation of sample error, percentage of the mean error to the sample mean, the root mean square error, percentage of the root mean square error to the sample mean, and the variance ratio (explained percentage variance, defined as one minus the ratio of pricing error variance to interest rate variance, in percentage)

Maturity Years	Mean	Std	ME%	RMSE	RMSE%	VR
1	-0.0246	0.3265	-0.621%	0.2211	5.594%	84.524%
2	0.0243	0.2843	0.548%	0.1710	3.863%	84.611%
3	0.0150	0.2583	0.312%	0.1364	2.837%	84.175%
4	-0.0064	0.2456	-0.125%	0.1319	2.596%	83.332%
5	-0.0270	0.2399	-0.511%	0.1413	2.674%	82.228%
7	-0.0509	0.2298	-0.913%	0.1592	2.856%	80.381%
10	-0.0567	0.2337	-0.970%	0.1877	3.214%	76.853%
15	-0.0004	0.2390	-0.007%	0.1992	3.264%	71.661%
30	-0.0325	0.2577	-0.520%	0.2364	3.791%	63.807%
Average	-0.0177	0.2572	-0.312%	0.1760	3.410%	79.063%

TABLE 4

Hazard Rate Estimation

This table reports the second-step parameter estimates that govern the dynamics of hazard rates for each firm given the parameters estimated in the first step. The estimation is based on a state space model (two interest rate factors and one hazard rate factor) using quasi-maximum likelihood method jointed with Unscented Kalman filter. The state equation and the measurement equation are specified in equation (20) and (21) respectively. The t-statistics are reported with corresponding parameter estimates.

Panel A: Top 30 Financial Firms

Name	Rating	$\hat{\alpha}_3$	$t(\hat{\alpha}_3)$	μ_3	$t(\mu_3)$	σ_3	$t(\sigma_3)$	λ_3	$t(\lambda_3)$	ρ_1	$t(\rho_1)$	ρ_2	$t(\rho_2)$
AMERICAN EXPRESS	AAA	0.645	5.35	0.115	1.52	1.198	2.17	-0.084	-1.01	-0.036	-5.66	-0.143	-2.68
AMERICAN INTERNATIONAL GROUP INC	AAA	0.533	4.94	0.087	4.79	1.090	3.98	-0.038	-3.81	0.278	1.48	0.056	4.19
GENERAL ELECTRIC CAPITAL CORP	AAA	0.611	2.24	0.086	3.94	1.165	5.99	-0.015	-7.52	-0.529	-2.22	-0.262	-1.05
ABN AMRO BANK N.V.	AA-	0.622	1.47	0.009	4.57	0.332	8.46	-0.029	-9.65	-0.188	-4.67	0.038	3.15
BANCO BILBAO VIZCAYA ARGENTARIA S.A.	AA-	0.571	2.08	0.056	1.98	1.569	2.97	-0.088	-1.71	-0.012	-0.52	-0.132	-3.84
BANK ONE CORPORATION	AA-	0.945	3.26	0.017	2.23	0.714	2.88	-0.060	-2.29	-0.480	-1.65	-0.006	-1.84
CITIGROUP INC.	AA-	1.633	6.34	0.023	1.77	0.200	1.76	-0.053	-5.02	0.174	1.65	0.198	8.27
DEUTSCHE BANK A.G.	AA-	1.223	6.08	0.017	9.43	0.021	1.73	-0.058	-1.26	-0.763	-6.71	-0.505	-1.56
FEDERAL NATIONAL MORTGAGE ASSOC.	AA-	1.646	1.78	0.022	4.77	0.364	7.25	-0.045	-4.67	-0.131	-2.36	-0.402	-4.22
SOCIETE GENERALE	AA-	0.583	3.07	0.011	1.60	0.160	5.89	-0.035	-1.99	0.062	6.84	0.059	5.69
BANCO SANTANDER CENTRAL HISPANO SA	A+	1.054	1.67	0.010	9.34	0.767	4.84	-0.038	-4.92	0.036	3.02	-0.065	-6.38
BANK OF AMERICA	A+	1.200	3.84	0.035	1.29	1.465	5.97	-0.075	-1.46	-0.851	-3.06	0.202	6.50
GOLDMAN SACHS	A+	1.108	5.97	0.053	3.29	1.005	5.22	-0.073	-2.77	-0.029	-1.17	-0.121	-4.01
JP MORGAN CHASE	A+	0.566	3.55	0.057	2.44	1.199	5.73	-0.079	-3.48	-0.035	-2.15	-0.141	-6.73
MERRILL LYNCH	A+	0.690	4.07	0.070	2.92	1.349	5.65	-0.074	-2.43	-0.031	-1.82	-0.109	-6.90
MORGAN STANLEY	A+	0.369	4.61	0.106	1.48	0.625	1.40	-0.019	-2.54	0.287	2.05	0.172	2.16
BEAR STEARNS CO	A	0.902	6.85	0.071	6.17	0.761	5.64	-0.071	-8.23	-0.030	-1.65	-0.112	-6.73
CIT GROUP INC	A	0.653	1.10	0.101	5.73	1.316	8.01	-0.049	-2.05	-0.236	-9.35	0.026	0.97
COUNTRYWIDE HOME LNS INC	A	0.991	6.39	0.087	1.84	0.641	5.30	-0.118	-1.57	-0.025	-4.50	-0.141	-2.26
HOUSEHOLD FINANCE CORPORATION	A	1.145	1.16	0.093	0.98	1.821	1.71	-0.317	-2.35	-0.034	-3.78	-0.016	-1.13
LEHMAN BROTHERS	A	1.336	2.70	0.119	3.37	2.408	6.17	-0.098	-2.14	-0.092	-3.01	-0.223	-9.19
BAYERISCHE HYPO-UND VEREINSBANK AG	A-	0.496	5.96	0.054	1.31	1.413	3.09	-0.078	-1.73	-0.034	-1.07	-0.121	-2.07
COMMERZBANK A.G.	A-	0.489	2.69	0.010	2.44	0.312	2.53	-0.004	-1.22	-0.486	-2.00	-0.455	-2.18
FLEETBOSTON FINANCIAL CORPORATION	A-	0.476	1.21	0.093	1.83	1.213	2.45	-0.036	-0.89	0.170	3.75	0.052	0.61
MUNICH RE	A-	0.668	5.67	0.146	5.81	1.779	3.53	-0.122	-4.20	-0.645	-2.62	-0.258	-1.32
MBNA AMERICA BANK, NA	BBB+	1.659	6.32	0.135	4.64	1.714	5.43	-0.081	-6.31	-0.555	-2.24	-0.106	-9.10
CAPITAL ONE BANK	BBB	0.349	3.68	0.375	1.38	0.499	6.21	-0.114	-0.91	0.221	2.65	-0.240	-1.75
MBNA CORPORATION	BBB	2.328	1.15	0.116	3.16	2.748	7.67	-0.183	-7.08	-0.116	-5.29	-0.221	-4.98

FORD MOTOR CREDIT COMPANY	BBB-	1.199	6.95	0.184	1.19	2.541	1.72	-0.165	-1.37	-0.889	-5.10	-0.416	-2.49
INTERNATIONAL LEASE FIN CORP	BB	2.496	6.25	0.134	1.19	3.145	3.74	-0.225	-2.71	-0.080	-2.73	-0.069	-0.87
Average		0.973	3.95	0.083	3.28	1.184	4.50	-0.084	-3.31	-0.169	-1.80	-0.115	-1.73

Panel B: Top 30 Industrial Firms

Name	Rating	$\hat{\alpha}_3$	$t(\hat{\alpha}_3)$	μ_3	$t(\mu_3)$	σ_3	$t(\sigma_3)$	λ_3	$t(\lambda_3)$	ρ_1	$t(\rho_1)$	ρ_2	$t(\rho_2)$
E. ON AG	AA-	0.468	5.35	0.086	1.52	1.392	2.17	-0.067	-1.01	-0.037	-0.57	-0.147	-2.68
SIEMENS AG	AA-	0.812	2.87	0.065	7.82	0.952	4.10	-0.071	-2.68	-0.028	-0.71	-0.144	-5.11
RWE AG	A+	0.479	9.12	0.081	2.01	1.391	2.16	-0.050	-0.50	-0.069	-1.44	-0.115	-1.45
VERIZON GLOBAL FUNDING	A+	1.882	6.14	0.106	6.08	2.760	3.56	-0.387	-1.44	-0.073	-0.89	-0.131	-5.12
BAYER AG	A	0.550	9.15	0.069	9.57	1.474	2.13	-0.052	-9.14	-0.034	-8.16	-0.160	-5.94
BOEING COMPANY	A	0.106	6.15	0.096	1.74	0.400	6.64	-0.109	-0.78	-0.334	-1.46	-0.269	-3.90
MCDONALD'S CORP	A	0.185	9.72	0.032	4.10	0.147	8.05	-0.114	-0.51	0.147	1.12	0.197	6.24
REUTERS GROUP PLC	A	1.938	3.37	0.225	5.61	3.665	7.27	-0.421	-1.15	-0.060	-1.98	-0.106	-3.60
PORTUGAL TELECOM INT'AL FINANCE BV	A-	0.722	2.66	0.074	5.52	0.886	3.65	-0.044	-1.54	-0.176	-9.63	-0.230	-8.95
VOLKSWAGEN AG	A-	0.637	1.20	0.089	1.75	1.384	2.43	-0.097	-4.28	-0.034	-1.13	-0.128	-4.76
WORLDCOM INC	A-	0.051	9.27	0.003	2.42	3.969	2.91	-0.046	-3.16	-0.150	-1.05	-0.076	-1.23
ALTRIA GROUP INC	BBB+	0.842	3.59	0.128	8.05	1.230	6.51	-0.067	-8.13	0.452	1.96	-0.145	-6.85
AOL TIME WARNER INC	BBB+	0.773	5.01	0.105	6.60	1.245	9.39	-0.163	-1.39	0.104	1.06	-0.054	-0.60
KONINKLIJKE (ROYAL) PHILIPS ELEC NV	BBB+	1.257	8.69	0.074	4.23	0.990	3.88	-0.091	-3.68	-0.026	-1.49	-0.145	-6.14
REPSOL S.A.	BBB+	0.608	3.62	0.087	2.62	1.083	6.43	-0.049	-1.80	-0.039	-1.42	-0.201	-9.09
WALT DISNEY COMPANY	BBB+	0.165	9.30	0.084	8.47	0.404	8.68	-0.161	-3.43	0.030	3.09	-0.208	-2.65
DAIMLERCHRYSLER AG	BBB	1.245	7.66	0.027	3.29	1.561	1.76	-0.123	-1.42	-0.734	-4.15	0.760	4.43
HEWLETT PACKARD CO	BBB	1.790	3.27	0.097	2.53	2.131	5.74	-0.116	-2.80	-0.040	-0.98	-0.271	-5.43
MOTOROLA INC	BBB	1.460	1.01	0.276	4.42	2.516	2.11	-0.334	-3.30	-0.039	-1.04	-0.224	-8.18
SEARS ROEBUCK ACCEPTANCE CORP	BBB	0.690	7.18	0.159	1.99	1.339	1.11	-0.088	-0.98	0.229	1.61	0.126	1.59
COMPUTER ASSOC INTL INC	BBB-	1.454	4.71	0.251	2.90	2.330	8.08	-0.647	-1.54	-0.100	-1.91	-0.037	-0.84
DELPHI	BBB-	0.266	1.09	0.264	1.36	0.440	1.27	-0.069	-3.29	0.081	1.29	0.010	2.41
EASTMAN KODAK COMPANY	BBB-	0.789	5.25	0.132	1.23	1.517	1.82	-0.073	-2.25	-0.003	-2.32	-0.026	-1.91
FEDERATED DEPARTMENT STORES INC	BBB-	3.059	2.67	0.131	1.11	3.687	4.14	-0.331	-9.88	-0.066	-0.90	-0.013	-1.32
SPRINT CORPORAT	BBB-	0.654	1.48	0.469	1.91	2.978	1.18	-0.105	-4.87	-0.850	-3.38	-0.129	-4.27
VIVENDI UNIVERSAL S.A	BBB-	0.120	2.24	1.435	7.85	1.553	3.28	-0.202	-1.64	0.373	1.13	-0.470	-2.57
VISTEON CORPORATION	BB+	1.094	3.30	0.191	5.72	2.028	5.37	-0.107	-2.36	-0.098	-3.32	-0.125	-3.64
TOYS "R" US INC	BB	1.239	6.49	0.127	4.35	2.084	3.63	-0.172	-5.55	-0.090	-4.89	-0.115	-2.24
FIAT SPA	BB-	0.897	1.03	0.066	6.55	1.308	1.09	-0.094	-6.75	-0.010	-1.22	-0.175	-1.21
GENERAL MOTOR	BB-	0.897	7.28	0.162	1.70	1.599	1.74	-0.077	-0.75	-0.387	-4.28	-0.233	-3.04
Average		0.904	5.00	0.173	4.17	1.681	4.08	-0.151	-3.07	-0.069	-1.57	-0.099	-2.93

TABLE 5

Summary Statistics of Pricing Errors on the CDS

This table reports the summary statistics of the pricing errors on the credit default swaps. We estimate our model (two interest rate factors and one hazard rate factor) with quasi-maximum likelihood method jointed with Unscented Kalman filter. We define the pricing error as the difference between the observed interest rate quotes and model-implied fair values. The columns titled with Mean, Std, RMSE%, VR denote the mean error, the standard deviation of sample error, percentage of the root mean square error to the sample mean, and the variance ratio (explained percentage variance, defined as one minus the ratio of pricing error variance to interest rate variance, in percentage)

Panel A: Financial Firms

Name	Rating	Mean	Std	RMSE%	VR
AMERICAN EXPRESS	AAA	-0.808	2.981	3.63%	93.57%
AMERICAN INTERNATIONAL GROUP INC	AAA	-1.605	4.207	7.76%	93.28%
GENERAL ELECTRIC CAPITAL CORP	AAA	-1.901	2.477	3.93%	98.39%
ABN AMRO BANK N.V.	AA-	-3.207	3.727	14.08%	85.38%
BANCO BILBAO VIZCAYA ARGENTARIA S.A.	AA-	-0.680	4.022	10.29%	90.47%
BANK ONE CORPORATION	AA-	-1.077	2.426	4.47%	94.28%
CITIGROUP INC.	AA-	-0.184	3.379	4.80%	95.78%
DEUTSCHE BANK A.G.	AA-	-2.788	2.462	10.25%	97.96%
FEDERAL NATIONAL MORTGAGE ASSO	AA-	-1.057	2.923	10.06%	55.21%
SOCIETE GENERALE	AA-	-5.117	5.852	17.62%	75.68%
BANCO SANTANDER CENTRAL HISPANO SA	A+	-1.265	3.780	6.67%	97.84%
BANK OF AMERICA	A+	-0.883	3.705	6.39%	86.42%
GOLDMAN SACHS	A+	-1.553	1.607	3.10%	98.54%
JP MORGAN CHASE	A+	-1.766	4.366	3.98%	95.31%
MERRILL LYNCH	A+	-0.273	9.330	4.81%	90.86%
MORGAN STANLEY	A+	-0.357	2.880	4.20%	97.78%
BEAR STEARNS CO	A	-1.134	3.952	3.81%	89.05%
CIT GROUP INC	A	-1.038	5.099	2.05%	99.56%
COUNTRYWIDE HOME LNS INC	A	0.123	8.154	3.62%	90.36%
HOUSEHOLD FINANCE CORPORATION	A	0.306	9.820	3.03%	99.66%
LEHMAN BROTHERS	A	-0.001	3.198	2.90%	94.86%
BAYERISCHE HYPO-UND VEREINSBANK AG	A-	6.439	24.161	11.06%	83.82%
COMMERZBANK A.G.	A-	5.027	31.502	14.30%	80.61%
FLEETBOSTON FINANCIAL CORPORATION	A-	-2.521	2.637	4.63%	99.02%
MUNICH RE	A-	-0.997	2.652	3.18%	97.98%
MBNA AMERICA BANK, NA	BBB+	0.566	5.386	1.67%	96.25%
CAPITAL ONE BANK	BBB	-1.227	10.763	1.97%	99.65%
MBNA CORPORATION	BBB	-0.433	8.056	1.92%	96.59%
FORD MOTOR CREDIT COMPANY	BBB-	-0.982	3.270	1.55%	99.95%
INTERNATIONAL LEASE FIN CORP	BB	-0.111	3.550	1.67%	94.72%
Average		-0.683	6.077	5.78%	92.29%

Panel B: Industrial Firms

Name	Rating	Mean	Std	RMSE%	VR
E. ON AG	AA-	-1.818	2.212	5.18%	98.85%
SIEMENS AG	AA-	-0.681	2.946	4.14%	95.38%
RWE AG	A+	-1.838	1.990	4.76%	99.52%
VERIZON GLOBAL FUNDING	A+	1.256	5.524	2.39%	99.41%
BAYER AG	A	-1.548	1.909	3.44%	99.87%
BOEING COMPANY	A	-5.921	7.716	8.30%	89.59%
MCDONALD'S CORP	A	-3.813	6.067	9.89%	92.14%
REUTERS GROUP PLC	A	0.012	4.520	1.39%	98.35%
PORTUGAL TELECOM INT'AL FINANCE BV	A-	-1.092	5.780	3.73%	96.99%
VOLKSWAGEN AG	A-	0.101	4.556	6.46%	93.36%
WORLDCOM INC	A-	-1.997	7.556	2.25%	99.77%
ALTRIA GROUP INC	BBB+	1.566	10.352	4.50%	98.15%
AOL TIME WARNER INC	BBB+	-1.563	4.409	2.61%	99.89%
KONINKLIJKE (ROYAL) PHILIPS ELEC NV	BBB+	-1.326	2.205	2.15%	99.48%
REPSOL S.A.	BBB+	-0.316	8.575	2.91%	99.45%
WALT DISNEY COMPANY	BBB+	6.631	20.540	8.74%	99.00%
DAIMLERCHRYSLER AG	BBB	-0.770	5.724	2.64%	97.05%
HEWLETT PACKARD CO	BBB	-1.113	3.489	2.77%	98.45%
MOTOROLA INC	BBB	4.338	13.344	2.48%	96.17%
SEARS ROEBUCK ACCEPTANCE CORP	BBB	1.217	7.995	4.98%	99.57%
COMPUTER ASSOC INTL INC	BBB-	5.206	36.164	3.21%	85.77%
DELPHI	BBB-	0.740	12.296	6.20%	93.38%
EASTMAN KODAK COMPANY	BBB-	5.400	31.343	5.02%	80.89%
FEDERATED DEPARTMENT STORES INC	BBB-	-0.398	4.105	1.98%	96.69%
SPRINT CORPORAT	BBB-	1.900	7.768	1.41%	99.84%
VIVENDI UNIVERSAL S.A	BBB-	6.631	20.540	4.88%	99.00%
VISTEON CORPORATION	BB+	2.231	11.279	3.36%	95.64%
TOYS "R" US INC	BB	3.392	19.450	3.73%	97.39%
FIAT SPA	BB-	-0.782	4.863	1.38%	99.93%
GENERAL MOTOR	BB-	0.867	6.848	4.27%	99.38%
Average		0.550	9.402	4.04%	96.61%

TABLE 6**Summary Statistics of Default Risk Premium**

This table reports the summary statistics of default risk premium for each firm. Default risk premium is calculated with the difference of model implied fair value of CDS under risk neutral parameters (market price of risk is zero) and estimated optimal parameters. Mean, Std, DRP% denote mean of default risk premium in basis points, the standard deviation of default risk premium, and percentage of average default risk premium to the mean of the corresponding credit default swap.

Panel A: Financial Firms

Name	Rating	Mean	Std	DRP%
AMERICAN EXPRESS	AAA	2.524	0.716	3.92%
AMERICAN INTERNATIONAL GROUP INC	AAA	2.647	1.360	5.24%
GENERAL ELECTRIC CAPITAL CORP	AAA	1.554	0.813	2.35%
ABN AMRO BANK N.V.	AA-	0.750	0.337	2.57%
BANCO BILBAO VIZCAYA ARGENTARIA S.A.	AA-	1.198	0.608	3.55%
BANK ONE CORPORATION	AA-	1.606	0.553	4.09%
CITIGROUP INC.	AA-	2.492	2.120	5.47%
DEUTSCHE BANK A.G.	AA-	1.644	0.984	4.21%
FEDERAL NATIONAL MORTGAGE ASSO	AA-	0.536	0.118	2.08%
SOCIETE GENERALE	AA-	0.934	0.506	3.14%
BANCO SANTANDER CENTRAL HISPANO SA	A+	1.294	0.677	2.44%
BANK OF AMERICA	A+	1.050	0.443	2.66%
GOLDMAN SACHS	A+	2.082	0.634	3.52%
JP MORGAN CHASE	A+	2.913	1.003	4.03%
MERRILL LYNCH	A+	1.171	0.828	1.64%
MORGAN STANLEY	A+	5.570	2.570	9.09%
BEAR STEARNS CO	A	2.731	0.710	4.06%
CIT GROUP INC	A	5.434	1.878	2.44%
COUNTRYWIDE HOME LNS INC	A	4.449	3.012	6.70%
HOUSEHOLD FINANCE CORPORATION	A	28.329	24.965	12.85%
LEHMAN BROTHERS	A	1.443	0.719	2.00%
BAYERISCHE HYPO-UND VEREINSBANK AG	A-	3.478	3.163	3.73%
COMMERZBANK A.G.	A-	3.883	0.506	4.41%
FLEETBOSTON FINANCIAL CORPORATION	A-	0.978	0.682	1.57%
MUNICH RE	A-	1.233	0.551	2.12%
MBNA AMERICA BANK, NA	BBB+	4.363	2.349	3.13%
CAPITAL ONE BANK	BBB	44.378	21.024	10.28%
MBNA CORPORATION	BBB	12.725	4.831	7.32%
FORD MOTOR CREDIT COMPANY	BBB-	13.473	10.727	5.95%
INTERNATIONAL LEASE FIN CORP	BB	0.105	0.299	0.08%
Average		5.232	2.990	4.22%

Panel B: Industrial Firms

Name	Rating	Mean	Std	DRP%
E. ON AG	AA-	1.454	0.870	2.52%
SIEMENS AG	AA-	6.408	2.910	10.53%
RWE AG	A+	3.300	1.762	4.96%
VERIZON GLOBAL FUNDING	A+	19.143	13.982	12.92%
BAYER AG	A	1.682	1.494	2.07%
BOEING COMPANY	A	8.597	3.040	11.14%
MCDONALD'S CORP	A	6.119	3.087	12.38%
REUTERS GROUP PLC	A	11.597	9.585	7.50%
PORTUGAL TELECOM INT'AL FINANCE BV	A-	3.106	1.220	2.95%
VOLKSWAGEN AG	A-	2.552	1.576	4.61%
WORLDCOM INC	A-	4.046	3.155	1.60%
ALTRIA GROUP INC	BBB+	4.275	3.011	3.26%
AOL TIME WARNER INC	BBB+	18.022	15.119	9.10%
KONINKLIJKE (ROYAL) PHILIPS ELEC NV	BBB+	5.141	2.155	4.89%
REPSOL S.A.	BBB+	6.885	4.463	3.39%
WALT DISNEY COMPANY	BBB+	14.274	6.874	15.69%
DAIMLERCHRYSLER AG	BBB	5.730	2.759	4.31%
HEWLETT PACKARD CO	BBB	3.470	1.992	3.56%
MOTOROLA INC	BBB	30.988	12.131	10.89%
SEARS ROEBUCK ACCEPTANCE CORP	BBB	7.792	5.963	4.21%
COMPUTER ASSOC INTL INC	BBB-	72.762	37.512	22.50%
DELPHI	BBB-	9.150	3.787	6.16%
EASTMAN KODAK COMPANY	BBB-	1.555	5.077	2.33%
FEDERATED DEPARTMENT STORES INC	BBB-	6.736	3.837	6.93%
SPRINT CORPORAT	BBB-	10.920	9.911	3.58%
VIVENDI UNIVERSAL S.A	BBB-	37.658	21.632	9.34%
VISTEON CORPORATION	BB+	7.767	3.479	4.14%
TOYS "R" US INC	BB	24.326	13.069	7.93%
FIAT SPA	BB-	36.281	9.733	5.03%
GENERAL MOTOR	BB-	5.658	3.423	3.26%
Average		12.580	6.954	6.79%

TABLE 7

Summary Statistics of Correlation

This table reports the empirical and model implied correlations among different time series.

$\rho_{r,d}$ represents model implied correlation between the instantaneous interest and hazard rates,

$\rho_{CDS,swap}$ represents the empirical correlation between the CDS spreads and the 5-year swap rates,

$\rho_{DRP,swap}$ represents the correlation between the model-implied default risk premiums and the 5-year swap rates, and finally

$\rho_{DRP,CDS}$ represents the model-implied correlation between the default risk premiums and the CDS spreads.

Panel A: Financial Firms

Name	Rating	$\rho_{r,d}$	$\rho_{CDS,swap}$	$\rho_{DRP,swap}$	$\rho_{DRP,CDS}$
AMERICAN EXPRESS	AAA	0.263	-0.012	-0.057	0.971
AMERICAN INTERNATIONAL GROUP INC	AAA	-0.674	-0.609	-0.677	0.959
GENERAL ELECTRIC CAPITAL CORP	AAA	-0.322	-0.649	-0.724	0.983
ABN AMRO BANK N.V.	AA-	-0.517	-0.604	-0.626	0.944
BANCO BILBAO VIZCAYA ARGENTARIA S.A.	AA-	-0.244	-0.467	-0.497	0.957
BANK ONE CORPORATION	AA-	0.210	-0.049	-0.232	0.895
CITIGROUP INC.	AA-	-0.584	-0.551	-0.571	0.938
DEUTSCHE BANK A.G.	AA-	-0.783	-0.748	-0.795	0.991
FEDERAL NATIONAL MORTGAGE ASSO	AA-	-0.388	-0.121	-0.522	0.792
SOCIETE GENERALE	AA-	-0.660	-0.528	-0.670	0.889
BANCO SANTANDER CENTRAL HISPANO SA	A+	-0.431	-0.656	-0.592	0.958
BANK OF AMERICA	A+	-0.317	-0.366	-0.271	0.929
GOLDMAN SACHS	A+	-0.534	-0.566	-0.546	0.984
JP MORGAN CHASE	A+	-0.440	-0.546	-0.557	0.991
MERRILL LYNCH	A+	-0.078	-0.420	-0.387	0.980
MORGAN STANLEY	A+	-0.228	-0.558	0.860	-0.818
BEAR STEARNS CO	A	0.045	0.128	-0.005	0.937
CIT GROUP INC	A	0.083	-0.177	-0.054	0.954
COUNTRYWIDE HOME LNS INC	A	-0.227	-0.151	-0.204	0.956
HOUSEHOLD FINANCE CORPORATION	A	-0.203	-0.302	-0.307	0.998
LEHMAN BROTHERS	A	-0.068	-0.278	-0.564	0.866
BAYERISCHE HYPO-UND VEREINSBANK AG	A-	-0.838	-0.769	-0.704	0.981
COMMERZBANK A.G.	A-	-0.127	-0.718	-0.670	0.889
FLEETBOSTON FINANCIAL CORPORATION	A-	-0.342	-0.668	-0.681	0.997
MUNICH RE	A-	0.316	-0.078	0.804	-0.323
MBNA AMERICA BANK, NA	BBB+	0.294	0.238	0.195	0.855
CAPITAL ONE BANK	BBB	-0.423	-0.337	-0.367	0.997
MBNA CORPORATION	BBB	-0.147	-0.457	-0.372	0.947
FORD MOTOR CREDIT COMPANY	BBB-	-0.860	-0.887	-0.891	0.998
INTERNATIONAL LEASE FIN CORP	BB	-0.541	-0.540	-0.808	0.834
Average		-0.292	-0.415	-0.383	0.841

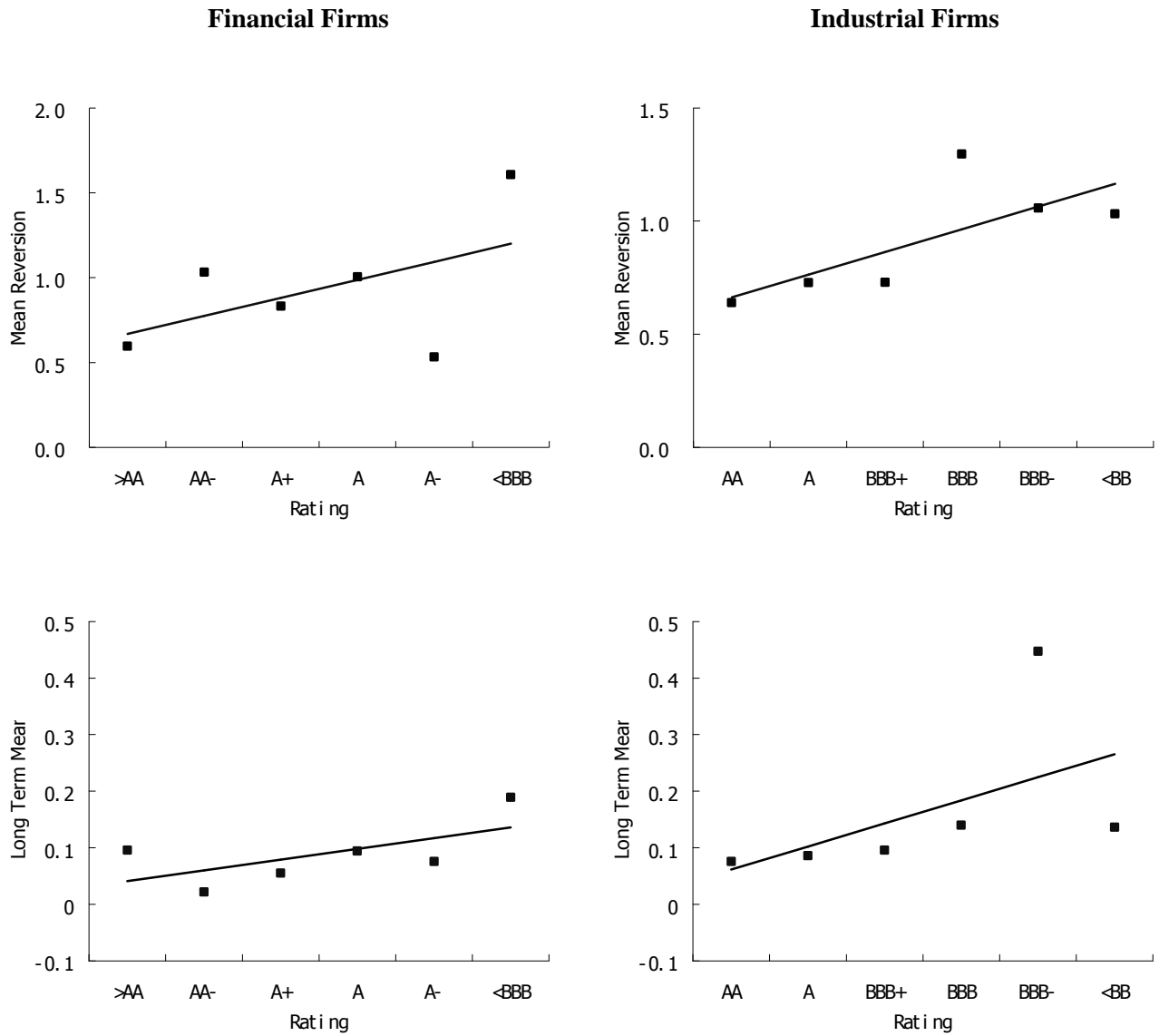
Panel B: Industrial Firms

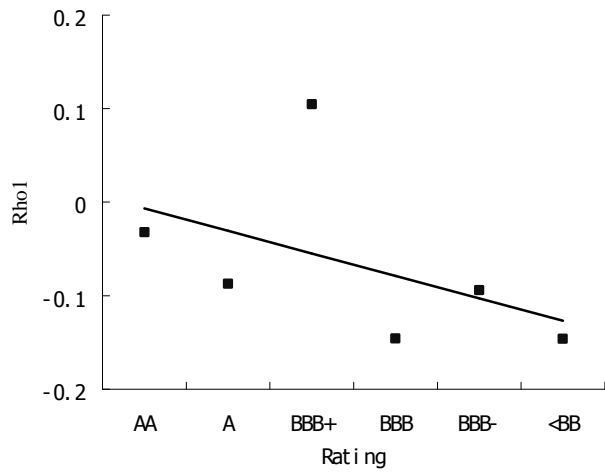
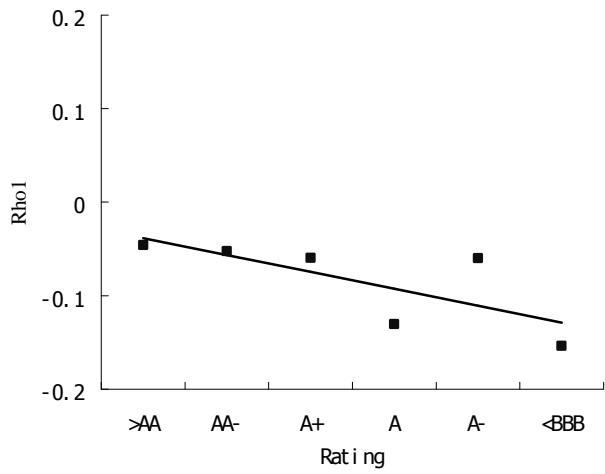
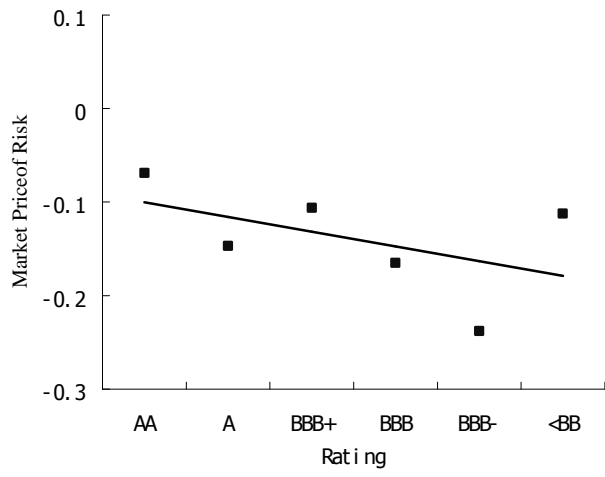
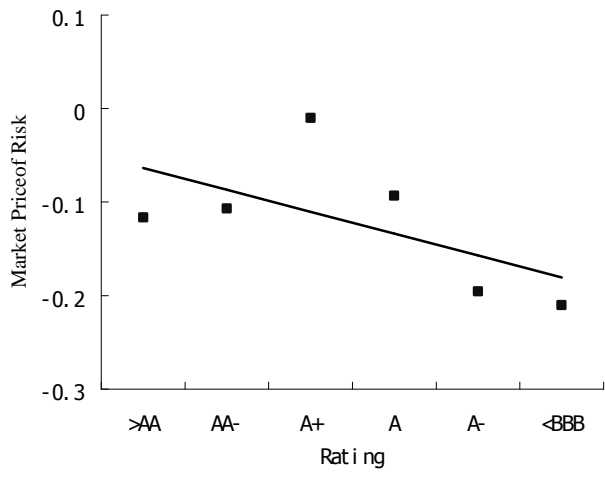
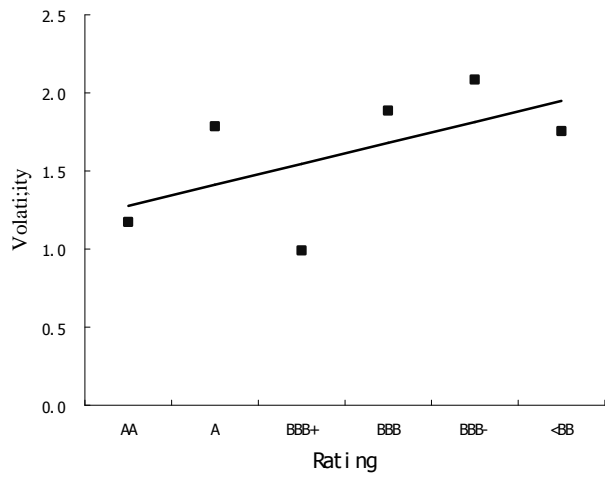
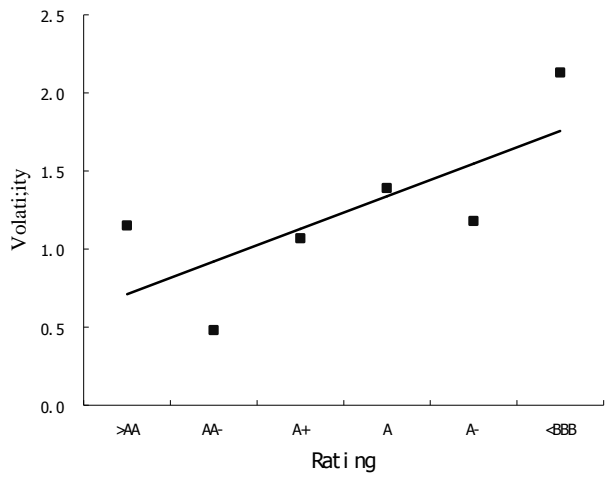
Name	Rating	$\rho_{r,d}$	$\rho_{CDS,swap}$	$\rho_{DRP,swap}$	$\rho_{DRP,CDS}$
E. ON AG	AA-	-0.021	-0.782	-0.773	0.992
SIEMENS AG	AA-	-0.674	-0.544	0.859	-0.787
RWE AG	A+	-0.907	-0.912	-0.927	0.993
VERIZON GLOBAL FUNDING	A+	-0.028	-0.223	-0.267	0.989
BAYER AG	A	-0.564	-0.802	-0.826	0.992
BOEING COMPANY	A	0.113	-0.443	-0.500	0.943
MCDONALD'S CORP	A	-0.607	-0.433	-0.469	0.679
REUTERS GROUP PLC	A	0.647	0.344	-0.711	0.232
PORTUGAL TELECOM INT'AL FINANCE BV	A-	0.173	0.148	0.181	0.981
VOLKSWAGEN AG	A-	-0.421	-0.818	-0.847	0.963
WORLDCOM INC	A-	-0.551	-0.273	-0.427	0.926
ALTRIA GROUP INC	BBB+	-0.604	-0.657	-0.655	0.993
AOL TIME WARNER INC	BBB+	-0.459	-0.509	-0.447	0.996
KONINKLIJKE (ROYAL) PHILIPS ELEC NV	BBB+	-0.579	-0.635	-0.613	0.992
REPSOL S.A.	BBB+	0.072	0.170	0.138	0.992
WALT DISNEY COMPANY	BBB+	-0.345	-0.568	-0.565	0.961
DAIMLERCHRYSLER AG	BBB	-0.354	-0.527	-0.618	0.949
HEWLETT PACKARD CO	BBB	-0.466	-0.476	-0.587	0.952
MOTOROLA INC	BBB	-0.472	-0.366	-0.534	0.820
SEARS ROEBUCK ACCEPTANCE CORP	BBB	-0.719	-0.791	-0.789	0.996
COMPUTER ASSOC INTL INC	BBB-	0.234	0.149	0.206	0.950
DELPHI	BBB-	-0.688	-0.600	-0.639	0.963
EASTMAN KODAK COMPANY	BBB-	0.260	0.241	0.135	0.954
FEDERATED DEPARTMENT STORES INC	BBB-	-0.731	-0.182	-0.824	0.152
SPRINT CORPORAT	BBB-	-0.433	-0.484	-0.434	0.974
VIVENDI UNIVERSAL S.A	BBB-	-0.480	-0.541	-0.500	0.993
VISTEON CORPORATION	BB+	-0.620	-0.739	-0.600	0.964
TOYS "R" US INC	BB	-0.456	-0.739	-0.511	0.933
FIAT SPA	BB-	-0.587	-0.629	-0.642	0.993
GENERAL MOTOR	BB-	-0.883	-0.933	-0.922	0.996
Average		-0.372	-0.452	-0.470	0.848

FIGURE 1

Parameter Estimates and Ratings

This figure plots the rating group mean of parameter estimates of CDS from the second-step estimation with the credit rating. The parameters are mean reversion, long-term mean, volatility, market price of risk, and the correlation between interest rate and hazard rate.





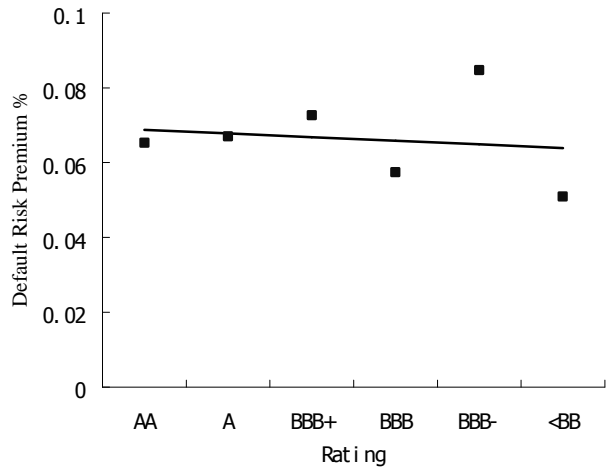
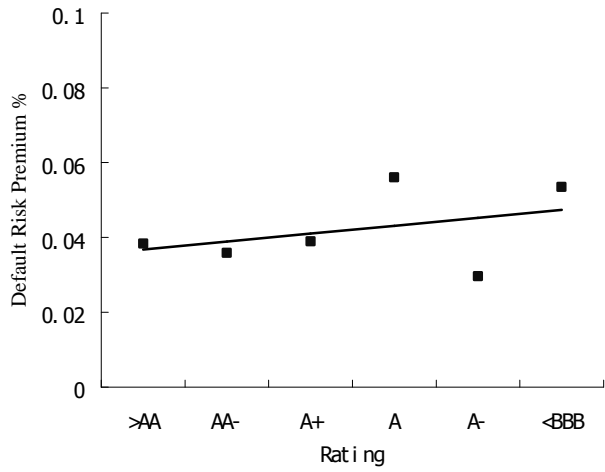
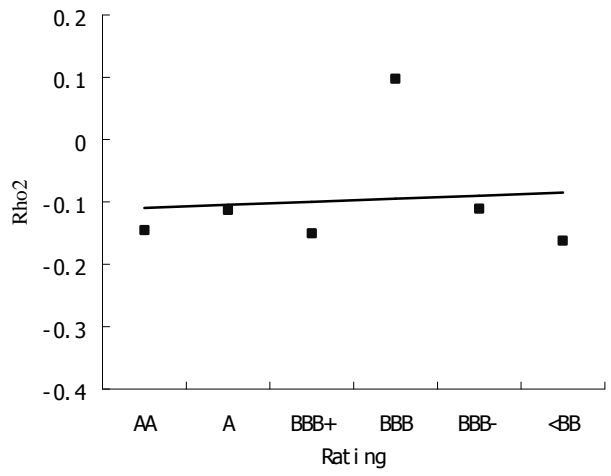
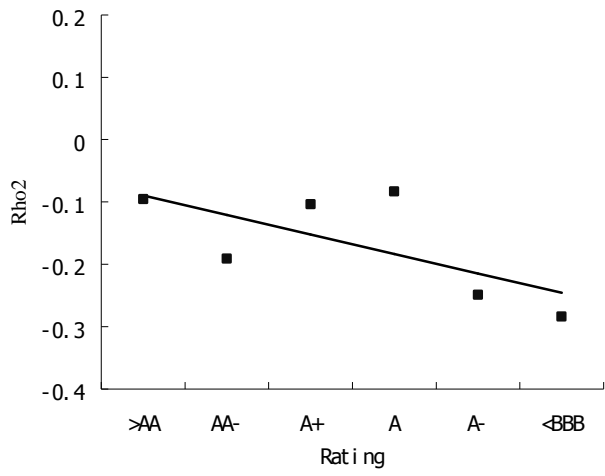


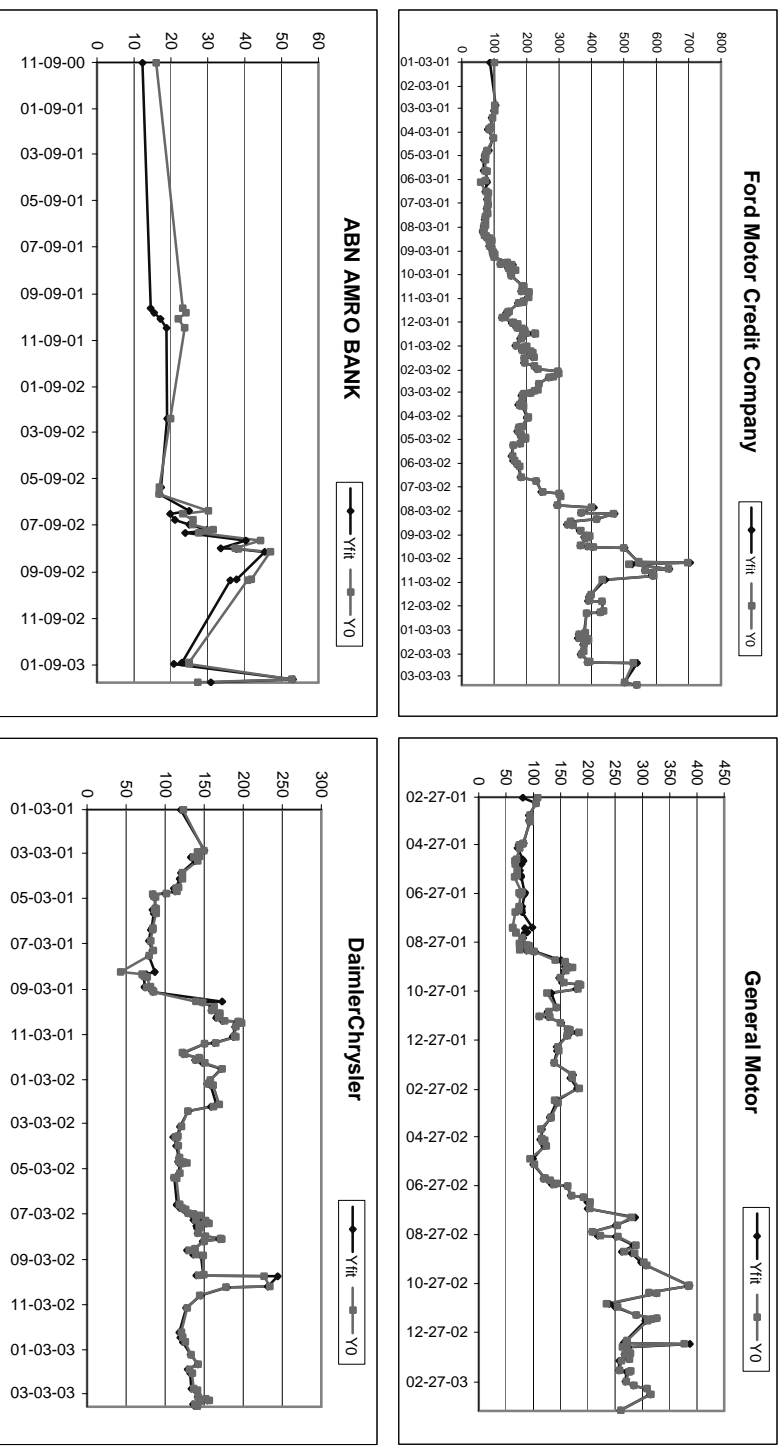
FIGURE 2

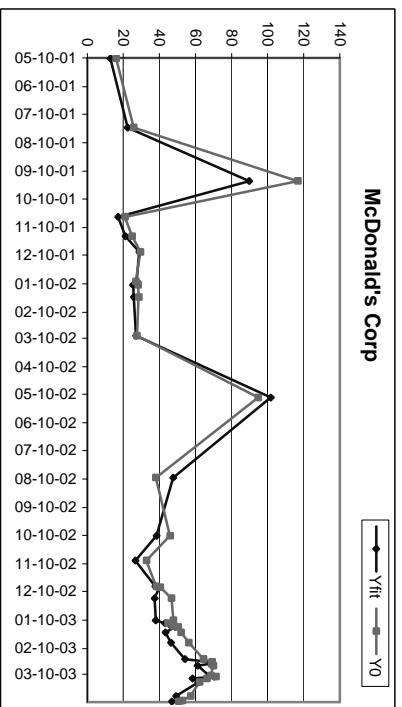
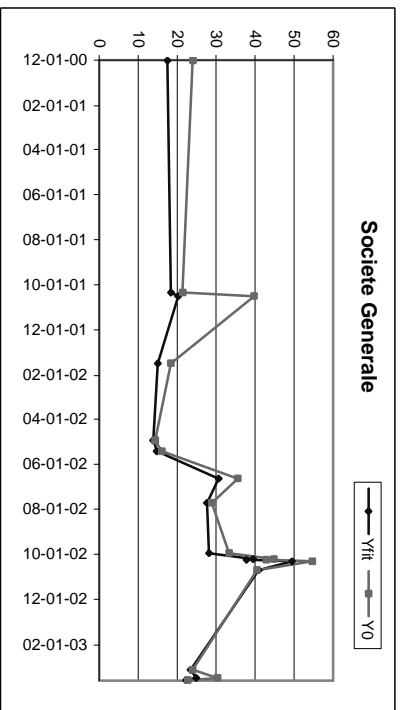
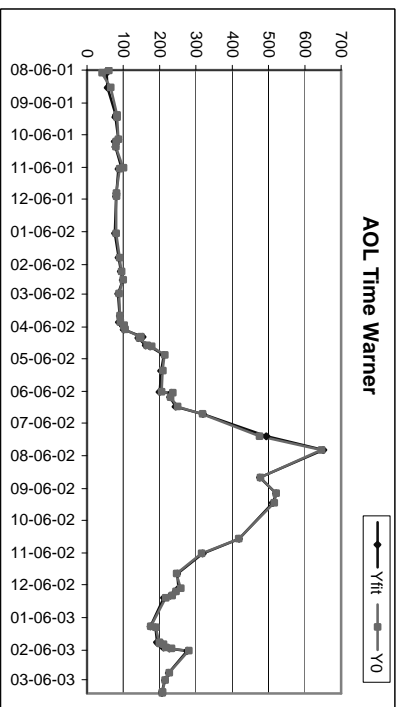
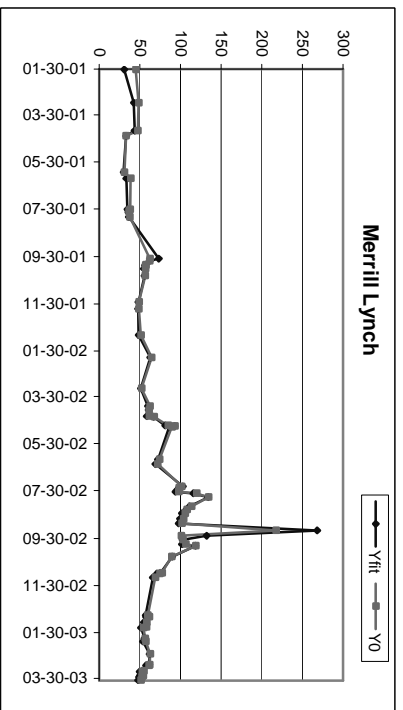
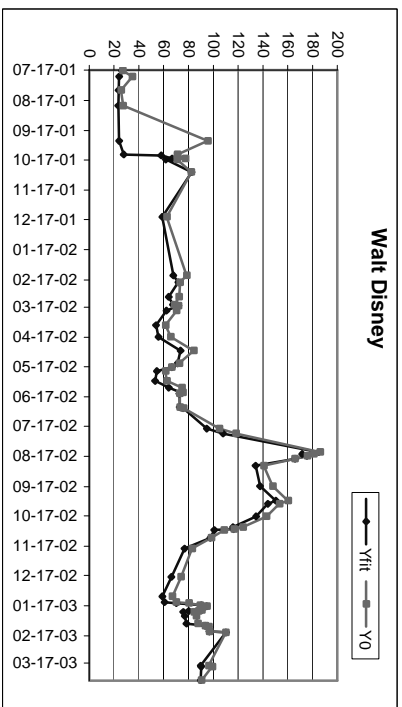
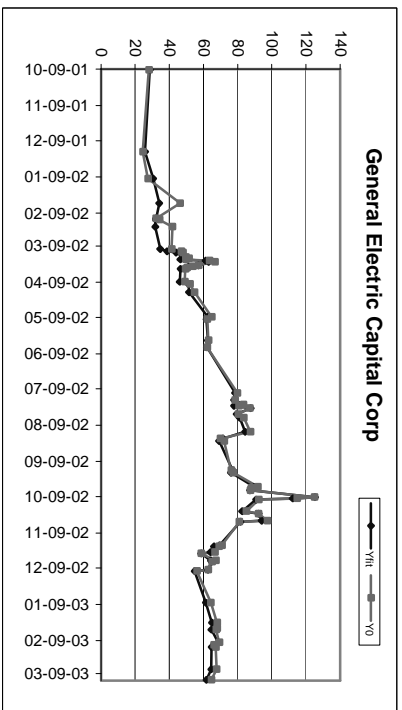
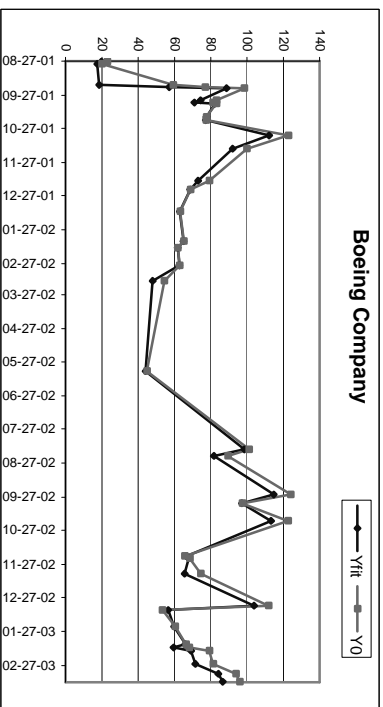
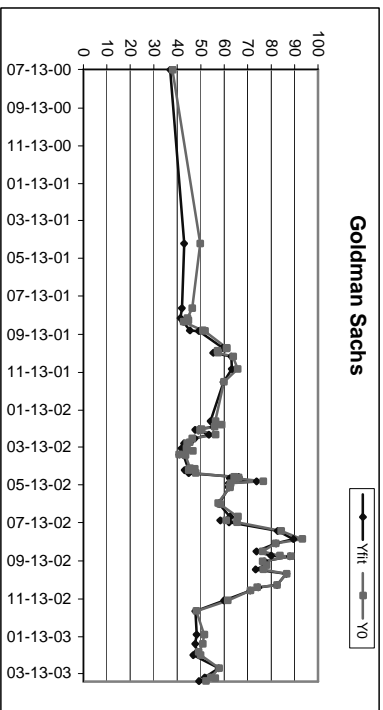
Time Series of Credit Default Swap Spread

This figure shows the time series of quoted spread (Y_0) and model implied spread (Y_{fit}) of specific firm. Data range from February 2000 to April 2003.

Financial Firms

Industrial Firms





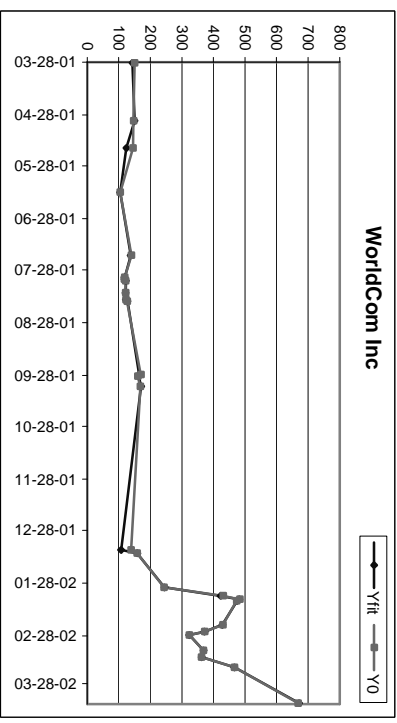
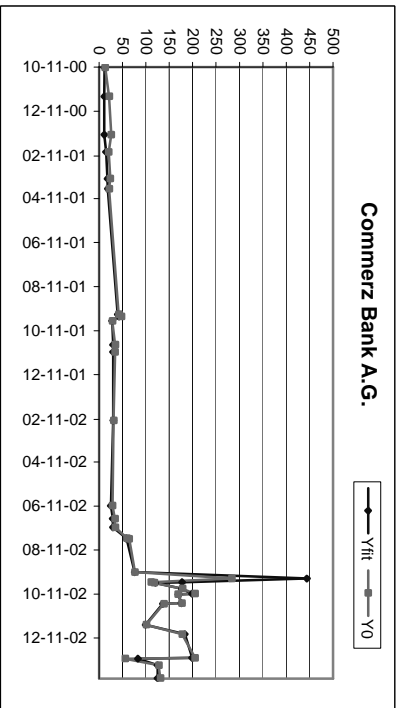
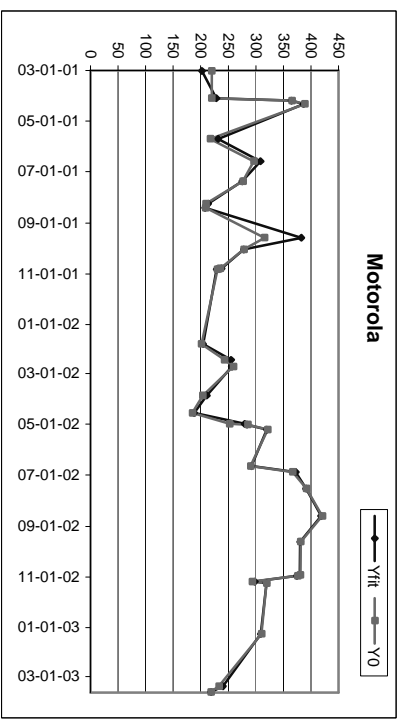
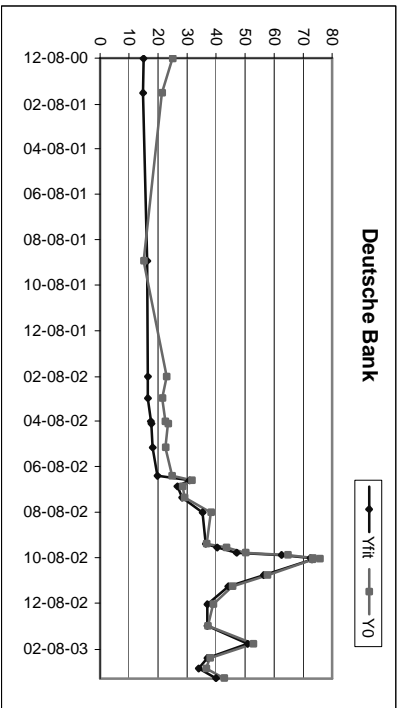
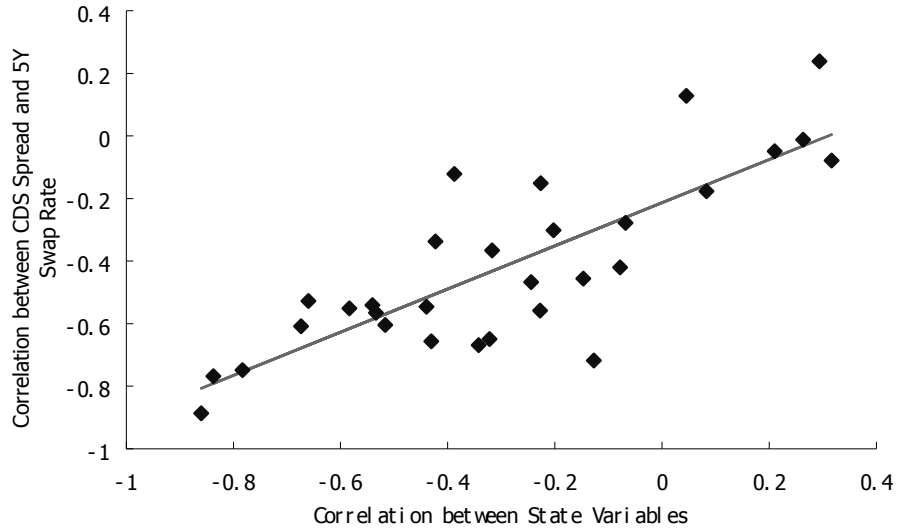


FIGURE 3

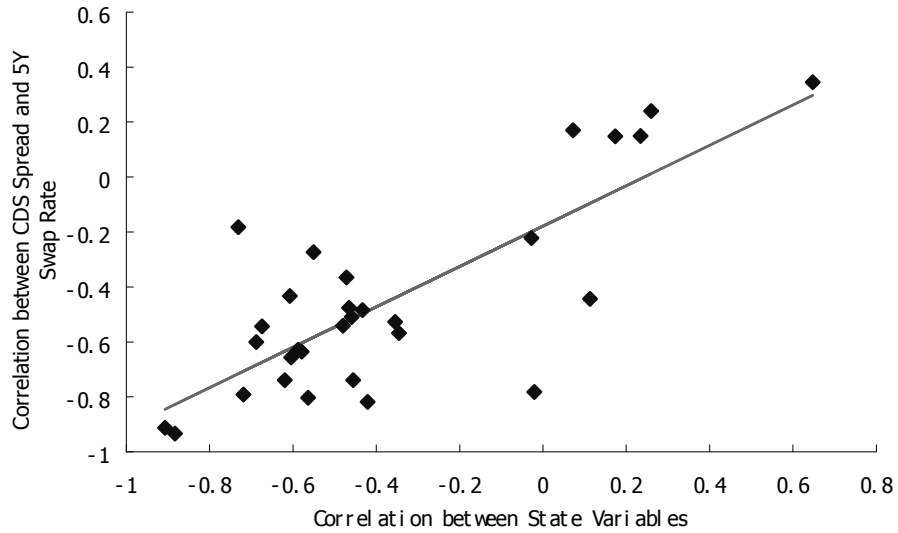
Comparison between Empirical and Model Implied Correlation

This figure compares the empirical correlation between CDS and 5-year swap rate with correlation of model implied state variables between the interest rate and hazard rate for each firm.

Financial Firms



Corporate Firms



VIII. Appendix

A. Proof of Proposition 1

In the proof for Propositions 1, as well as for Propositions 2 and 3, we use the following notation

$$[1] \quad dX(t) = AX(t)dt + \Omega dW(t)$$

where $W(t)$ are independent Wiener processes.

The risk-free discount factor (zero-coupon bond price) is the solution to the following risk-neutral expectation:

$$[2] \quad \begin{aligned} P(0, t) &= E_Y \left[\exp \left(- \int_0^t r(u) du \right) \right] \\ &= E_Y \left[\exp \left(- \int_0^t \sum_{i=1}^n \beta_i(u) + Y^\top(u)Y(u) du \right) \right] \\ &= \exp \left(- \int_0^t \sum_{i=1}^n \beta_i(u) \right) \tilde{E}_Y \left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \exp \left(- \int_0^t Y^\top(u)Y(u) du \right) \right] \end{aligned}$$

where

[3]

$$\begin{aligned} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} &= \exp \left(- \int_0^t Y^\top(u)\Gamma(u, t)\Omega dW(u) - \frac{1}{2} \int_0^t Y^\top(u)\Gamma(u, t)\Omega\Omega^\top\Gamma^\top(u)Y(u) du \right) \\ &= \exp \left(- \int_0^t (Y^\top(u)\Gamma(u, t)dY(u) - X^\top(u)\Gamma(u, t)AY(u)du) - \frac{1}{2} \int_0^t Y^\top(u)\Gamma(u)\Omega\Omega^\top\Gamma^\top(u, t)X(u)du \right) \end{aligned}$$

The rest of the steps are similar to the derivation in the following sub-section.

B. Proof of Proposition 2

The risky zero-coupon bond price is:

$$\begin{aligned}
V(0, t) &= E_{Y,Z} \left[\exp \left(- \int_0^t r(u) + h(u) du \right) \right] \\
&= E_{Y,Z} \left[\exp \left(- \int_0^t \left(\sum_{i=1}^{\ell} \beta_i(u) + \sum_{i=1}^n b_i(u) + Y^\top(u)Y(u) + Z^\top(u)Z(u) \right) du \right) \right] \\
[4] \quad &= \exp \left(- \int_0^t \left(\sum_{i=1}^{\ell} \beta_i(u) + \sum_{i=1}^n b_i(u) \right) du \right) E_{Y,Z} \left[\exp \left(- \int_0^t (Y^\top(u)Y(u) + Z^\top(u)Z(u)) du \right) \right] \\
&= \exp \left(- \int_0^t \left(\sum_{i=1}^{\ell} \beta_i(u) + \sum_{i=1}^n b_i(u) \right) du \right) \tilde{E}_{Y,Z} \left[\frac{d\mathbb{P}}{d\mathbb{P}} \exp \left(- \int_0^t (Y^\top(u)Y(u) + Z^\top(u)Z(u)) du \right) \right]
\end{aligned}$$

where

[5]

$$\begin{aligned}
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} &= \exp \left(- \int_0^t X^\top(u)\Gamma(u, t)\Omega dW(u) - \frac{1}{2} \int_0^t X^\top(u)\Gamma(u, t)\Omega\Omega^\top\Gamma^\top(u)X(u)du \right) \\
&= \exp \left(- \int_0^t (X^\top(u)\Gamma(u, t)dX(u) - X^\top(u)\Gamma(u, t)AX(u)du) - \frac{1}{2} \int_0^t X^\top(u)\Gamma(u)\Omega\Omega^\top\Gamma^\top(u, t)X(u)du \right)
\end{aligned}$$

and $\tilde{\mathbb{P}}$ is a new probability measure (Riccati-implied) we construct to obtain a free degree of freedom.

Applying Ito's Lemma on the first term of the last line of equation [4], we get,

$$\begin{aligned}
d[X^\top(u)\Gamma(u, t)X(u)] &= X^\top(u)\Gamma(u, t)dX(u) + d[X^\top(u)]\Gamma(u, t)X(u) + X^\top(u)\Gamma'(u, t)X(u)du \\
&\quad + d[X^\top(u)]\Gamma(u, t)d[X(u)] + X^\top(u)\Gamma^\top(u, t)dud[X(u)] + d[X^\top(u)]\Gamma^\top(u, t)duX(u) \\
&= 2X^\top(u)\Gamma(u, t)dX(u) + X^\top(u)\Gamma^\top(u, t)X(u)du + tr[\Omega^\top\Gamma(u, t)\Omega]du
\end{aligned}$$

where $\Gamma'(u, t) = \frac{d\Gamma(u, t)}{du}$ for $u \leq t$. Therefore,

$$[6] \quad X^\top(u)\Gamma(u, t)dX(u) = \frac{1}{2}d[X^\top(u)\Gamma(u, t)X(u)] - \frac{1}{2}X^\top(u)\Gamma^\top(u, t)X(u)du - \frac{1}{2}tr[\Omega^\top\Gamma(u, t)\Omega]du$$

Substituting [6] back into [5], we get,

$$\begin{aligned}
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} &= \exp \left[- \frac{1}{2}X^\top(t)\Gamma(t, t)X(t) + \frac{1}{2}X^\top(0)\Gamma(0, t)X(0) \right. \\
[7] \quad &\quad + \int_0^t \frac{1}{2}X^\top(u)\Gamma'(u, t)X(u)du + \int_0^t \frac{1}{2}tr[\Omega^\top\Gamma(u, t)\Omega]du \\
&\quad \left. + \int_0^t X^\top(u)\Gamma(u, t)AX(u)du - \frac{1}{2} \int_0^t X^\top(u)\Gamma(u, t)\Omega\Omega^\top\Gamma^\top(u, t)X(u)du \right]
\end{aligned}$$

Substituting the above result into [4]

$$\begin{aligned}
V(0, t) &= \tilde{E}_{y,z} \left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \exp \left(- \int_0^t \left(\sum_{i=1}^{\ell} b_i(u) + \sum_{i=1}^n \beta_i(u) + Y^\top(u)Y(u) + Z^\top(u)Z(u) \right) du \right) \right] \\
&= \tilde{E}_{y,z} \left[\exp \left\{ \frac{1}{2} X^\top(t)\Gamma(t,t)X(t) - \frac{1}{2} X^\top(0)\Gamma(0,t)X(0) \right. \right. \\
&\quad - \int_0^t X^\top(u) \left[\frac{\Gamma'(u,t)}{2} + \Gamma(u)A - \frac{\Gamma(u,t)\Omega\Omega^\top\Gamma(u,t)}{2} + I \right] X(u) du \\
&\quad \left. \left. - \int_0^t \left(\sum_{i=1}^{\ell} b_i(u) + \sum_{i=1}^n \beta_i(u) + \frac{1}{2} \text{tr}[\Omega^\top\Gamma(u,t)\Omega] \right) du \right\} \right]
\end{aligned}
\tag{8}$$

Note that the $\Gamma(u, t)$ is arbitrary, which is the free degree of freedom we have in the model. Hence, we can set it so that the quadratic term of the second line of the final expression of [8] is equal to zero. Hence, we arrive at the following ordinary differential equation (ODE):

$$\Gamma'(u, t) + \Gamma(u, t)A + A\Gamma(u, t) - \Gamma(u, t)\Sigma\Sigma^\top\Gamma(u, t) + 2I = 0
\tag{9}$$

where $\Gamma'(u, t) = \frac{d\Gamma(u, t)}{du}$ for $u \leq t$ and I is an identity matrix. Equation [9] is known as the Riccati equation.

Given that $\Gamma(u, t)$ is arbitrary, for simplicity, we let the terminal condition $\Gamma(t, t) = 0$ and let:

$$N(u) = \begin{bmatrix} n_1(u-t) & n_2(u-t) \\ n_3(u-t) & n_4(u-t) \end{bmatrix} = \exp \left(\begin{bmatrix} -A & -2I \\ -\Omega\Omega^\top & A \end{bmatrix} (u-t) \right)
\tag{10}$$

Then the solution to [9] is:

$$\Gamma(u, t) = n_2(u-t)[n_4(u-t)]^{-1}
\tag{11}$$

Using the result of [9], [8] can be simplified to the following equation

$$\begin{aligned}
V(0, t) &= E_{Y,Z} \left[\exp \left(- \int_0^t (r(u) + h(u)) du \right) \right] \\
&= \exp \left(- \frac{1}{2} X^\top(0)\Gamma(0,t)X(0) \right) \exp \left(- \frac{1}{2} \int_0^t \sum_{i=1}^{\ell} \beta_i(u) + \sum_{i=1}^n b_i(u) + \text{tr}[\Omega^\top\Gamma(u,t)\Omega] du \right)
\end{aligned}
\tag{12}$$

which completes the proof.

C. Proof of Proposition 3

The payoff of a CDS for a unit notional is 1 minus the recovery rate at the default time τ :

$$[13] \quad G(\tau) = 1_{\tau < T}(1 - \xi(\tau))$$

where $1_{\{\cdot\}}$ is an indicator function and $\xi(\cdot)$ is the recovery rate which can be random. For the sake of simplicity and without loss of generality, we assume a constant recovery rate throughout this paper. As a result, the price of a CDS is the discounted risk-neutral expectation of $G(\tau)$:

$$[14]$$

$$\begin{aligned} D(0, T) &= \mathbb{E}_{Y, Z, \tau} \left[\exp \left(- \int_0^\tau r(u) du \right) G(\tau) \right] = \mathbb{E}_{Y, Z, \tau} \left[1_{\tau < T} \exp \left(- \int_0^\tau r(u) du \right) \right] (1 - \xi) \\ &= \mathbb{E}_{Y, Z} \left[\mathbb{E}_\tau \left[1_{\tau < T} \exp \left(- \int_0^\tau \sum_{i=1}^n \beta_i(u) + Y^\top(u) Y(u) du \right) \middle| Y(\tau), z(\tau) \right] \right] (1 - \xi) \\ &= \mathbb{E}_{Y, Z} \left[\int_0^T \left(\sum_{i=1}^\ell b_i(\tau) + Z^\top(\tau) Z(\tau) \right) \exp \left(- \int_0^\tau \left(\sum_{i=1}^\ell b_i(u) + Z^\top(u) Z(u) \right) du \right) \right. \\ &\quad \left. \exp \left(- \int_0^\tau \left(\sum_{i=1}^n \beta_i(u) + Y^\top(u) Y(u) \right) du \right) d\tau \right] (1 - \xi) \\ &= (1 - \xi) \int_0^T \mathbb{E}_{Y, Z} \left[\left(\sum_{i=1}^\ell b_i(\tau) + Z^\top(\tau) Z(\tau) \right) \exp \left(- \int_0^\tau \left(\sum_{i=1}^\ell b_i(u) + \sum_{i=1}^n \beta_i(u) + Y^\top(u) Y(u) + Z^\top(u) Z(u) \right) du \right) \right] d\tau \\ &= (1 - \xi) \int_0^T \Delta(0, \tau) d\tau \end{aligned}$$

Note that on the second line, the conditional expectation is conditional on the entire evolution till default time: $y(\tau), z(\tau)$. The third line is simply the explicit use of the exponential density function. The last line exchanges the expectation and integral operation.

Under the $\tilde{\mathbb{P}}$ -measure (but up to time τ):

$$[15] \quad d\tilde{W}(t) = dW(t) + \Omega^\top \Gamma(t, \tau) X(t) dt$$

We define the following matrices:

$$[16] \quad \Phi(t) = X^\top(t) X(t)$$

Applying Ito's lemma, then taking the mean, that is $\Psi = E[\Phi]$, we obtain:

$$[17] \quad \begin{aligned} d\Psi(t) &= [\Psi(t)C^\top(t) + C(t)\Psi(t) + \Omega\Omega^\top]dt \\ \Psi'(t) - \Psi(t)C^\top(t) - C(t)\Psi(t) - \Omega\Omega^\top &= 0 \end{aligned}$$

where $\Psi'(t) = \frac{d\Psi(t)}{dt}$ and $C(t) = A - \Omega\Omega^\top\Gamma(t, \tau)$

Therefore,

$$Z^\top(t)Z(t) = \Psi(t) \begin{bmatrix} 0_{n \times 1} \\ 1_{\ell \times 1} \end{bmatrix}$$

The final result is

[18]

$$\begin{aligned} \Delta(0, \tau) &= \underset{Y, Z}{E} \left[\left(\sum_{i=1}^{\ell} b_i(\tau) + Z^\top(\tau)Z(\tau) \right) \exp \left(- \int_0^\tau \left(\sum_{i=1}^{\ell} b_i(u) + \sum_{i=1}^n \beta_i(u) + Y^\top(u)Y(u) + Z^\top(u)Z(u) \right) du \right) \right] \\ &= \underset{Y, Z}{\tilde{E}} \left[\frac{d\mathbb{P}}{d\mathbb{P}} \left(\sum_{i=1}^{\ell} b_i(\tau) + Z^\top(\tau)Z(\tau) \right) \exp \left(- \int_0^\tau \left(\sum_{i=1}^{\ell} b_i(u) + \sum_{i=1}^n \beta_i(u) + Y^\top(u)Y(u) + Z^\top(u)Z(u) \right) du \right) \right] \\ &= \left(\sum_{i=1}^{\ell} b_i(\tau) + \Psi(t) \begin{bmatrix} 0_{n \times 1} \\ 1_{\ell \times 1} \end{bmatrix} \right) \times \\ &\quad \exp \left\{ - \frac{1}{2} X^\top(0)\Gamma(0, \tau)X(0) - \int_0^\tau \left(\sum_{i=1}^{\ell} b_i(u) + \sum_{i=1}^n \beta_i(u) + \frac{1}{2} tr[\Omega^\top\Gamma(u, \tau)\Omega] \right) du \right\} \end{aligned}$$

and $D(0, T) = (1 - \xi) \int_0^T \Delta(0, \tau) d\tau$, which completes the proof.

D. Forward Measure

Given a general process for the interest rate as:

$$[19] \quad dr(t) = \alpha(r, t)dt + \sigma(r, t)dW(t)$$

and the risk-free zero-coupon bond price as:

$$[20] \quad P(0, t) = E \left[\exp \left(- \int_0^t r(u)du \right) \right],$$

the change of T -maturity forward measure is given by:

$$\begin{aligned}
dr(t) &= \alpha(r, t)dt + \sigma(r, t)dW(t) \\
[21] \quad &= \left(\alpha(r, t) - \frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial r} \sigma^2(r, t) \right) dt + \sigma(r, t)d\hat{W}^{(T)}
\end{aligned}$$

The Radon-Nikodym Derivative for this change of measure is:

$$[22] \quad \eta = \frac{\exp\left(-\int_0^t r(u)du\right)}{E\left[\exp\left(-\int_0^t r(u)du\right)\right]} = \frac{1}{P(0, t)} \exp\left(-\int_0^t r(u)du\right)$$

Note that under the risk-neutral measure, we have

$$[23] \quad \frac{dP(t, T)}{P(t, T)} = r(t)dt + \sigma_p(t)dW(t)$$

for all $T > t$. Taking log of the log normal stochastic process:

$$\begin{aligned}
0 &= \ln P(T, T) = \ln P(0, T) + \int_0^T r(u)du - \frac{\sigma_p^2(u)}{2} \int_0^T du + \int_0^T \sigma_p(u)dW(u) \\
[24] \quad \frac{\exp\left(-\int_0^T r(u)du\right)}{P(0, T)} &= \exp\left(\int_0^T \sigma_p(u)dW(u) - \int_0^T \frac{\sigma_p^2(u)}{2} du\right)
\end{aligned}$$

and hence the Girsanov transformation is defined over:

$$[25] \quad dW(t) = d\hat{W}^{(T)}(t) + \sigma_p(t)dt$$

where by Ito's lemma, we have

$$[26] \quad \sigma_p(t) = \frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial r} \sigma(r, t)$$

E. Computational Efficiency

To demonstrate both the ease with which our model can be used and its computational efficiency, we degenerate our model to the standard CIR model. In the simplest setting where $\beta(t) = b(t) = 0$, $\rho = 0$, and a single factor for both the risk-free rate and the hazard rate, the quadratic model for a risky bond becomes the product of two CIR models under $\mu = 0$:

$$[27] \quad P(0, t) = \Phi(0, t) e^{-r^{(0)}G_1(0, T) - h^{(0)}G_2(0, T)}$$

where

$$\Phi(0, t) = \left[\frac{2\zeta_1 e^{(\alpha_1 + \zeta_1)T/2}}{(\alpha_1 + \zeta_1)(e^{\zeta_1 T} - 1) + 2\zeta_1} \right]^{1/2} \left[\frac{2\zeta_2 e^{(\alpha_2 + \zeta_2)T/2}}{(\alpha_2 + \zeta_2)(e^{\zeta_2 T} - 1) + 2\zeta_2} \right]^{1/2}$$

$$G_i(0, t) = \frac{2(e^{\zeta_i t} - 1)}{(\alpha_i + \zeta_i)(e^{\zeta_i t} - 1) + 2\zeta_i}$$

$$\zeta_i = \sqrt{\alpha_i^2 + 2\sigma_i^2}$$

Setting the other parameter values as follows:

	y	z
α	0.5	0.3
σ	0.2	0.3
initial value	0.1	0.2
T	1	

where $r = y^2$ is the instantaneous risk-free rate and $h = z^2$ is the instantaneous hazard rate, we can compute the CIR price of the zero-coupon bond to be 0.9452. In our model, the solution to the Riccati equation requires the following matrix given in [10] in Section VIII B:

$$N(0, 1) = \begin{bmatrix} 0.7881 & 0 & \boxed{2.0276} & 0 \\ 0 & 0.8822 & \boxed{0} & \boxed{2.0223} \\ 0.0101 & 0 & \boxed{1.2950} & 0 \\ 0 & 0.0228 & \boxed{0} & \boxed{1.1856} \end{bmatrix}$$

which implies the two following matrices:

$$n_2(0, 1) = \begin{bmatrix} 2.0276 & 0 \\ 0 & 2.0226 \end{bmatrix} \text{ and } n_4(0, 1) = \begin{bmatrix} 1.2950 & 0 \\ 0 & 1.1856 \end{bmatrix}$$

Substituting the above result back in [11] to obtain the solution to the Riccati equation:

$$\Gamma(0, 1) = n_2(0, 1)[n_4(0, 1)]^{-1} = \begin{bmatrix} 1.5657 & 0 \\ 0 & 1.7059 \end{bmatrix}$$

This allows us to compute the first and the second terms in (6) as follows:

$$\exp\left(-\frac{1}{2} X^\top(0)\Gamma(0, 1)X(0)\right) = 0.9589 \text{ and } \exp\left(-\frac{1}{2} \int_0^1 \text{tr}[\Omega^\top \Gamma(u, 1)\Omega]du\right) = 0.9587$$

which yields the bond price as 0.9452, identical to the CIR model.

Note that our model is very fast. Although it does require a numerical integration in the last step as seen above, this integration is no different from the Gaussian integration used in many Black-Scholes-like models.

Since our model is a special case of the Duffie-Pan-Singleton model and they provide a numerical solution to their model, we next compare the computational efficiency between our Black-Scholes-like analytical solution and their model. In general, ODEs resulting from the affine models are very quick to solve. In a single-factor setting and valuing only a few options, solving ODEs is fast enough not to be detected by human eyes. However, if we compute a large number of option prices with many factors, such as in a situation where the model needs to be calibrated to a large number of market prices, the efficiency of the ODEs starts to deteriorate exponentially. In the table that follows, we compute 125 bond prices (all having one year to maturity) with various parameter combinations:

	Duffie-Pan-Singleton solution				Our Model	CIR
ODE error tolerance	eps=1e-3	eps=1e-6	eps=1e-9	eps=1e-12		
average time (sec)	0.004	0.00825	0.0285	0.155	0.007	0.006
pricing error	0.01246660	0.00044120	0.00032090	0.00032170	0.00000000	

The computations were performed using Dell 700M, 1.6 giga herz CPU, Pentium M and SAS IML. ODE solvers and numerical integration use standard SAS libraries.

We see two problems with using the ODE solution. First, the pricing error will not approach 0 as we increase the numerical accuracy. As we see it, when the ODE tolerance level is increased from 10^{-9} to 10^{-12} , the pricing error can no longer decrease. On the contrary, our model produces an accuracy level at 10^{-8} . Second, the computation efficiency of our model is comparable to the ODE solution that has errors at 10s of basis points.

F. Unscented Kalman Filter

The Kalman filter and its various extensions belong to the state-space estimation regime and are based on a pair of state propagation equations and measurement equations. In our applications with Gaussian state variables, we can write the state-propagation generally as:

$$[28] \quad X_t = A + \Phi X_{t-1} + \sqrt{Q} \varepsilon_t$$

We can also write the measurement equation in a generic form,

$$[29] \quad Y_t = h(X_t; \Theta) + e_t$$

where Y_t denotes the observed series at time t and $h(X_t; \Theta)$ denotes their corresponding fair values based on the model as a function of the state vector X_t and model parameters Θ . The last term e_t denotes the measurement error of the series at time t .

Let \bar{X}_t , $\bar{\Sigma}_t$, \bar{Y}_t , and \bar{V}_t denote the time $t-1$ *ex ante* forecasts of time t values of the state vector, the covariance of the state vector, the measurement series, and the covariance of the measurement series, respectively. Let \hat{X}_t and $\hat{\Sigma}_t$ denote the *ex post* update, or filtering, on the state vector and its covariance at the time t based on observations (Y_t) at time t . In the case of the linear measurement equation,

$$[30] \quad Y_t = HX_t + e_t$$

The Kalman filter provides the most of the efficient updates. The *ex ante* predictions are:

$$[31] \quad \begin{aligned} \bar{X}_t &= A + \Phi \hat{X}_{t-1} \\ \bar{\Sigma}_t &= \Phi \hat{\Sigma}_{t-1} \Phi^T + Q \\ \bar{Y}_t &= H\bar{X}_t \\ \bar{V}_t &= H\bar{\Sigma}_t H^T + R \end{aligned}$$

and the *ex post* filtering updates are:

$$[32] \quad \begin{aligned} \hat{X}_{t+1} &= \bar{X}_{t+1} + K_{t+1}(Y_{t+1} - \bar{Y}_{t+1}) \\ \hat{\Sigma}_{t+1} &= \bar{\Sigma}_{t+1} - K_{t+1} \bar{V}_{t+1} K_{t+1}^T \end{aligned}$$

where K_{t+1} is the Kalman gain, given by:

$$[33] \quad K_{t+1} = \bar{\Sigma}_{t+1} H^T (\bar{V}_{t+1})^{-1}$$

However, in our application, the measurement equation in (16) and (21) is nonlinear. Traditionally, nonlinearity is often handled by the extended Kalman filter (EKF), which approximates the nonlinear measurement equation with a linear expansion, evaluated at the predicted states:

$$[34] \quad Y_t \approx H(\bar{X}_t; \Theta) X_t + e_t$$

where

$$H(\bar{X}_t; \Theta) = \left. \frac{\partial h(\bar{X}_t; \Theta)}{\partial X_t} \right|_{X_t = \bar{X}_t}$$

The rest of prediction and updates follow equation [31] and [32]. We note that the extended Kalman filter uses only one point (the conditional mean) from the prior filtering density for the prediction and filtering updates.

In contrast, the unscented Kalman filter applied in this paper uses a set of points that are designed to also match higher moments. Let $p = 3$ be the number of states and $\delta > 0$ be a control parameter. Let A_i be the i -th column of a matrix A . A set of $2p + 1$ sigma vectors χ_i are generated according to the following equations:

$$[35] \quad \begin{aligned} \chi_{t,0} &= \hat{X}_t \\ \chi_{t,i} &= \hat{X}_t \pm \sqrt{(p + \delta)(\hat{V}_t + Q)}_j, j = 1, \dots, p; i = 1, \dots, 2p \end{aligned}$$

with corresponding weights w_i given by

$$[36] \quad w_0 = \frac{\delta}{(p + \delta)}, \quad w_i = \frac{1}{2(p + \delta)}, \quad i = 1, \dots, 2p$$

We can regard these sigma vectors as forming a discrete distribution with w_i as the corresponding probabilities. Then we can verify that the mean, covariance, skewness, and

kurtosis of this distribution are $\hat{X}_t, \hat{V}_t + Q, 0$, and $p + \delta$, respectively. Given the sigma points, the prediction steps are given by

$$\begin{aligned}
 \bar{X}_{t+1} &= \sum_{i=0}^{2p} w_i (\Phi_{\chi_{t,i}}) \\
 \bar{\Sigma}_{t+1} &= \sum_{i=0}^{2p} w_i (\Phi_{\chi_{t,i}} - \bar{X}_{t+1})(\Phi_{\chi_{t,i}} - \bar{X}_{t+1})^T \\
 \bar{Y}_{t+1} &= \sum_{i=0}^{2p} w_i h(\Phi_{\chi_{t,i}}; \Theta) \\
 \bar{V}_{t+1} &= \sum_{i=0}^{2p} w_i (h(\Phi_{\chi_{t,i}}; \Theta) - \bar{Y}_{t+1})(h(\Phi_{\chi_{t,i}}; \Theta) - \bar{Y}_{t+1})^T + R
 \end{aligned}
 \tag{37}$$

and the filtering updates are given by

$$\begin{aligned}
 \hat{X}_{t+1} &= \bar{X}_{t+1} + K_{t+1}(Y_{t+1} - \bar{Y}_{t+1}) \\
 \hat{\Sigma}_{t+1} &= \bar{\Sigma}_{t+1} - K_{t+1} \bar{V}_{t+1} K_{t+1}^T
 \end{aligned}
 \tag{38}$$

with the Kalman gain defined as

$$K_{t+1} = S_{t+1} (\bar{V}_{t+1})^{-1} = \sum_{i=0}^{2p} w_i (\Phi_{\chi_{t,i}} - \bar{X}_{t+1}) [h(\Phi_{\chi_{t,i}}) - \bar{Y}_{t+1}]^T (\bar{V}_{t+1})^{-1}
 \tag{39}$$

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